

Optimal Pricing of Information

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Abstract

A decision maker looks to take an active action (e.g., purchase some goods or make an investment). The payoff of this active action depends on his own private type as well as a random and unknown *state of nature*. To decide between this active action and another passive action, which always leads to a safe constant utility, the decision maker may purchase information from an *information seller*. The seller can access the realized state of nature, and this information is useful for the decision maker (i.e., the *information buyer*) to better estimate his payoff from the active action.

We study the seller's problem of designing a revenue-optimal pricing scheme to sell her information to the buyer. Suppose the buyer's private type and the state of nature are drawn from two independent distributions, we fully characterize the optimal pricing mechanism for the seller in closed form. Specifically, under a natural linearity assumption of the buyer payoff function, we show that an optimal pricing mechanism is the *threshold mechanism* which charges each buyer type some upfront payment and then reveals whether the realized state is above some threshold or below it. The payment and the threshold are generally different for different buyer types, and are carefully tailored to accommodate the different amount of risks each buyer type can take. The proof of our results relies on novel techniques and concepts, such as upper/lower virtual values and their mixtures, which may be of independent interest.

1 Introduction

In numerous situations, a decision maker wishes to take an active move but is uncertain about its outcome and payoff. Such active moves range from financial decisions of investing a stock or startup to daily life decisions of purchasing a house or used car, from macro-level enterprise decisions of developing a new product to micro-level decisions of approving a loan applicant or displaying online ads to a particular Internet user. In these situations, the decision maker’s payoff for the active move relies on uncertain information regarding, e.g., potential of the invested company, quality of the house, popularity of the new product, credit of the loan applicant, etc. Certainly, the decision maker typically also has a passive backup option of not making the move, in which case he obtains a safe utility without any risk. To decide between the *active* and *passive* action, the decision maker can turn to an information seller who can access more accurate information about the uncertainties and thus help to better estimate the payoff for his action. Given the usefulness of the seller’s information to the decision maker, the seller can make a profit from how much her information helped to improve utilities of the decision maker, i.e., the information buyer.

This paper studies how a monopoly information *seller* (she) described above can design an optimal pricing mechanism to sell her information to an information *buyer* (he) randomly drawn from some population. The buyer, a decision maker, needs to take one of two actions. The *active* action results in a payoff $v(q, t)$ where t captures the buyer’s private type and the *state of nature* q summarizes the payoff-relevant uncertainty unknown to the buyer. The q and t are independent random variables and are both supported on *continuous* sets. The passive backup action for the buyer always results in the same utility, normalized to 0, regardless of q, t . While the buyer and seller both know the distribution of q , the seller is more informed and can additionally observe the *realized* the state. With the information of q , the seller would like to design an optimal pricing scheme to sell her information to the buyer, whose type is drawn from a known distribution.

The described problem setup above is a very fundamental monopoly pricing problem. However, different from the classic pricing problem for goods, here we are looking for an optimal pricing scheme for information. Despite bearing a similar structure, these two pricing problems turns out to differ significantly. For example, when selling goods, physical or digital, the seller’s allocation rule can simply be described by a probability of giving out the goods. However, it is unclear here how the seller should “pass” her information to the buyer. Second, and more importantly, in selling goods, any individually rational buyer should participate in the mechanism as long as their expected utility is at least 0. However, in our setup, without participating in the mechanism, the buyer may already have positive utility from his active decisions. An individually rational buyer would participate in the mechanism only when his utility will become even higher. This key difference turns out to render standard mechanism design techniques inapplicable here (more details will be discussed later). This is also evidenced by our characterization of the optimal mechanism for the pricing of information, which is significantly different and more intricate than the optimal pricing of goods.

Main Result: Optimal Mechanism for Pricing Information. We characterize the optimal mechanism for the above information pricing problem, within the general class of mechanisms that can be described as a generic interactive protocol introduced by Babaioff et al. [2].¹ Specifically, assuming buyer’s value function is linear and monotone non-decreasing in t , i.e., $v(q, t) = v_1(q)[t + \rho(q)]$ for some $v_1(q) \geq 0$, we show that the optimal mechanism always admits a simple format — any buyer is incentivized to report his true type t (i.e., the mechanism is incentive compatible); the seller then charges the buyer p_t and, afterwards, reveals whether the realized state q satisfies $\rho(q) \geq \theta_t$ or not for some carefully chosen threshold θ_t . We thus call such a mechanism *threshold mechanism*. The thresholds and payments are different for different buyer types, and are carefully designed to accommodate the amount of risks each buyer type can take. We will

¹This class of mechanisms is quite general and includes all possible ways that the buyer may sequentially reveal information and ask for charges.

fully characterize the threshold and payment in the optimal mechanism. They turn out to have a somewhat intricate format and relies on novel concepts of *upper* and *lower* virtual values as well as their mixtures. As a byproduct, we also exhibit several interesting properties of the optimal mechanism. For example, the optimal mechanism will charge any two buyer types the same amount if the information revealed to them is the same. So the optimal mechanism will not discriminate buyer types. Moreover, we also provide clean characterizations about the monotonicity properties about the buyer’s *payment*, *utility*, and *surplus* as a function of buyer type. It turns out that they may increase or decrease, depending on the setup.

We make two remarks. First, we note that threshold mechanisms are actually ubiquitous in reality. In many application domains, we will need to pay for doing some inspections or tests for some entity, and then receive an outcome about whether it passed the inspection or not. These are precisely threshold mechanisms. Viewed this way, our results characterize the optimal threshold and payment for a buyer drawn from a random population. Second, since both q, t are in continuous space in our setup, therefore typical variable optimization based approaches (e.g., linear programming as used in many previous works [2, 7, 10]) would not apply to our setting. Our analysis is based on optimizing *functional variables*. The fortunate aspect, though, is that such functional optimization allows us to derive more structural optimal mechanisms, in a similar spirit to Myerson’s seminal work in optimal auction design [17].

Related Works. The study of markets for information has attracted extensive recent research interests. For a comprehensive overview of the progress in this field, we refer interested readers to a very recent survey by Bergemann and Bonatti [3]. This paper adopts the mechanism design approach and looks to design the optimal mechanism (within some design space) that maximizes the seller’s revenue. Next we discuss the recent works that are most relevant to us. Note that, the welfare maximization problem in selling information is relatively straightforward since the seller can simply reveal full information. Therefore, optimality in all our discussions always mean the optimality of *revenue*.

A starting point of our paper is the work by Babaioff et al. [2] who study information selling from a mechanism design perspective. They also consider monopoly pricing setup with one information seller and a buyer who relies on the seller’s information to improve his decision making. They consider mechanisms within a very general design space captured by *generic interactive protocols* [2] (see our descriptions in Section 2). These mechanisms include all possible ways the seller can sequentially reveal information via signaling schemes and ask for charges along the way. Within this space, Babaioff et al. establish a revelation-principle-type result and show that the seller can always use a somewhat *direct* mechanism to maximize revenue which uses a single signaling scheme and asks for charges before or after sending the signal, depending on the setup. Built upon this characterization, they also develop linear program formulations to compute the optimal mechanism in polynomial time. Our work utilizes the revelation principle of [2] but goes significantly beyond to further develop closed-form optimal mechanisms for selling information.

Recent works by Chen et al. [10] and Cai and Velekcas [7] also adopted algorithmic approaches to compute the optimal mechanisms in polynomial time for a budgeted buyer or in the setup with multiple buyers. Different from all these algorithmic approaches [2, 10, 7], our results are analytical. The optimal mechanisms we developed can be explicitly described and thus reveals more structural insights about the optimal mechanism. This choice of using analytical approaches is also somewhat necessary in our setup since agents have continuous utility functions and typical variable optimization approaches are not applicable to our setting. Bergemann et al. [4] also exhibit structural properties about the optimal mechanism within a special menu-based class of mechanisms. In their setting, each buyer type may have different beliefs about the state of nature whereas, like [2, 10], we assume all buyers share the same prior beliefs in our model. In their more challenging model, Bergemann et al. show that an analytical solution can be characterized when there are two buyer types or only two buyer actions and two states, whereas in general only partial properties about the optimal mechanism can be derived.

Our work is also relevant to the recent rich body of works on information design, a.k.a., Bayesian per-

suasion [15]. We refer curious readers to a few recent surveys by Bergemann and Morris [6], by Kamenica [14] and a survey from the algorithmic perspective by Dughmi [12]. Most relevant to ours is the persuasion problem with a privately informed receiver [16]. However, different from the sender in the persuasion problem who has a specific utility function, the seller in our model does not have any utility function except the transfer she received from the buyer. The optimal mechanism we obtain is also not comparable to the optimal signaling scheme characterized by Kolotilin et al. [16]. Notably, the case of binary receiver actions (or binary buyer actions) is an important special case of information design and has attracted significant attention in previous works [8, 9, 13, 1, 16]. Candogan [9] shows that threshold signaling scheme could be optimal when, e.g., the receiver utility is linear in payoff parameters as well the action he takes. Finally, Daskalakis et al. [11] study the joint design of signaling schemes and auction mechanisms. Though it is a very different setup from ours, the signaling scheme in their model can be viewed as a way to extract more revenue from bidders.

2 Model and Problem Formulation

2.1 Problem Setup

Motivated by various applications of quality testing, we consider the following optimal pricing problem of information between an information Seller (she) and an information Buyer (he). The buyer is a decision maker who faces one of two actions: a *passive* action 0 and an *active* action 1. Buyer obtains an uncertain payoff $v(q, t)$ for the active action 1 where $q \in Q$ is a random *state of nature* (unknown to the buyer) and $t \in T$ is the buyer's private type. Buyer's utility for the passive action 0 is always 0, irrespective of his type and the state of nature. So the passive action serves as a backup option for the buyer. For example, if Buyer is a potential purchaser of some goods with uncertain quality (e.g., a house or a used car), the passive action 0 corresponds to *not purchase* in which case Buyer has no gain neither loss, whereas the active action 1 corresponds to *purchase* in which Buyer's utility depends on the quality q of the goods as well as how much he values the goods, i.e., the private type t .

Both t and q are modeled as random variables that are independently distributed according to the cumulative distribution functions (CDF) $F(t)$ and $G(q)$, respectively. The buyer's type $t \in \mathbb{R}$ is a real value and supported on $T = [t_1, t_2]$. The state of nature q is supported on a general measurable set Q and *does not* need to be a real value. Such an abstract representation of q is used to accommodate applications where q may include the features relevant to Buyer's decisions (e.g., the brand and production year of a used car). Throughout the paper, we assume both $F(t)$ and $G(q)$ are differentiable with corresponding probability density functions (PDF) $f(t)$ and $g(q)$, though our analyses and results also extend to the non-differentiable situations.²

Both $F(t)$ and $G(q)$ are public knowledge. However, the realized q can be observed by the information seller. We study Seller's problem of designing a *revenue-optimal* pricing scheme to sell her private information about q to Buyer. Note that the buyer's private type t is only known to the buyer. Therefore, Seller will have to incentivize the buyer to report his type t before deciding how to reveal information to the buyer and how much to charge. Indeed, had the seller known the buyer's type t , the seller's optimal pricing mechanism should be revealing full information and then charges Buyer the *value of information* [4]: $\int_{q \in Q} \max\{0, v(q, t)\} g(q) dq - \max\{0, \int_{q \in Q} v(q, t) g(q) dq\}$.

We assume Buyer's utility function $v(q, t)$ is linear and monotone *non-decreasing* in the buyer's type t for any $q \in Q$. Consequently, there exists function $v_1(q) \geq 0$ and $\rho(q)$ such that

$$v(q, t) = v_1(q)(t + \rho(q)). \quad (1)$$

Since q is a random variable, $\rho(q)$ also has a probability distribution. For expositional simplicity, we assume the distribution of ρ does not have point masses, i.e., the probability measure of the set $\{q \mid \rho(q) = d\}$ for

²In this case, the density functions are called *distributions* or *generalized functions*.

any d is 0. Similar analysis applies to the general case where $\rho(q)$ contains point masses, except that the optimal mechanism may need some randomization in that case.

With slight abuse of notation, let $v(t)$ denote the buyer's expected utility for action 1 under his prior beliefs about q , namely, when no information is purchased. That is,

$$\text{Buyer's initial utility of action 1: } v(t) = \int_{q \in Q} v(q, t)g(q)dq. \quad (2)$$

Note that our setup above is similar to persuasion of a privately informed receiver by Kolotilin et al. [16]. The key difference is that in our model, Seller has no utility function and only cares about the transfer from Buyer (i.e., the revenue) whereas in [16] the sender has a particular utility function. Moreover, our buyer utility function in Equation (1) strictly generalizes the receiver's utility in [16] which assumes linearity in *both* the receiver's type t and the state of nature q .

2.2 The Revelation Principle

To maximize revenue, Seller can design arbitrary mechanisms with possibly multiple rounds of interactions. We restrict our design space to include all possible mechanisms that can be expressed as a finite extensive-form game where each node in the game tree is one of the following three types: (1) *transfer node* which is associated with a (possibly negative) transfer to the seller and has a single child node; (2) *seller node* which associates each state of nature q with a distribution prescribing the probabilities of moving to its children nodes; (3) *buyer node* which has arbitrary buyer actions and an arbitrary set of children. This general space of mechanisms is also referred to as the *generic interactive protocol* [2] and includes all possible ways that Seller may sequentially reveal partial information to Buyer and ask for transfers alone the way.

The space of all such possible mechanisms appears enormous. Fortunately, similar to classic mechanism design [17], the setting of information pricing also admits the revelation principle as shown in [2, 4, 10]. To describe the space of the so-called *direct* mechanisms, it suffices to introduce the notion of *signaling* scheme, which formalizes the way that Seller reveals information to Buyer. Formally, given a set of possible signals Σ , a signaling scheme $\pi : Q \rightarrow \Delta_\Sigma$ is a mapping from the state of nature q to a distribution over the signals in Σ . Such a signaling scheme can be mathematically described by $\{\pi(\sigma; q)\}_{q \in Q, \sigma \in \Sigma}$ where $\pi(\sigma; q)$ is the probability of sending signal σ conditioned on state q .³ Given signal σ , the buyer infers posterior probability about any state q via a standard Bayes updates:

$$\Pr(q|\sigma) = \frac{\pi(\sigma; q) \cdot g(q)}{\int_{q' \in Q} \pi(\sigma; q') \cdot g(q')dq'} = \frac{\pi(\sigma; q) \cdot g(q)}{\mathbf{E}_{q' \sim G}[\pi(\sigma; q')].} \quad (3)$$

Consequently, conditioned on signal σ , a buyer of type t has expected utility $\int_{q \in Q} v(q, t) \Pr(q|\sigma)dq$ for the active action 1, and will take the active action if and only if when $\int_{q \in Q} v(q, t) \Pr(q|\sigma)dq \geq 0$.

The following revelation principle is the starting point of our design of the optimal mechanism.

Lemma 1. *[Revelation Principle [2, 4, 10]] There always exists a revenue-maximizing mechanism for the above setting which consists of a menu $\{\pi_t, p_t\}_{t \in T}$ such that: (1) π_t is a signaling scheme for buyer type t which uses at most two signals $\{\sigma_0, \sigma_1\}$, resulting in the best buyer action 1 and 0, respectively; (2) p_t is the payment from the buyer of type t .*

This mechanism with one-round buyer-seller interaction proceeds in order as follows:

1. Seller announces the mechanism $\{\pi_t, p_t\}_{t \in T}$;
2. Buyer type t is realized;

³Such a signaling scheme is sometimes also referred to as an *experiments* [16, 4].

3. Buyer reports a type $t' \in T$ to Seller (not necessarily the true t), and is charged payment $p_{t'}$;
4. The state of nature q is realized to Seller, who then samples a signal according to signaling scheme $\pi_{t'}$ and sends the signal to Buyer.

Moreover, the mechanism without loss of generality is incentive compatible and individually rational.

Like classic mechanism design, incentive compatibility (IC) means reporting true type t is an optimal strategy for Buyer, and individual rationality (IR) means participating the mechanism is no worse for Buyer than not participating. Babaioff et al. [2] proved Lemma 1 in the form of general signaling schemes and later results [4, 10] further simplify it and show that it suffices to restrict to signaling schemes with n signals where n is the number of buyer actions ($n = 2$ in our case).

Our next result further simplifies the design space and shows that we can without loss of generality restrict to mechanisms with non-negative payment for every type t . Though this is natural to expect, we point out that this result does not trivially hold — in fact, it was shown that an optimal single-round mechanism may sometimes have to involve *negative* payments when buyer type q and state t are correlated [2]. The proof of this technical lemma is deferred to Appendix B.

Lemma 2. [Non-Negative Payments] There exists an optimal IC and IR mechanism in which $p_t \geq 0$ for all $t \in T$.

2.3 Formulation of the Optimal Pricing Problem

We refer to the mechanisms characterized by Lemma 1 as *direct* mechanisms. Given the revelation principle, we can without loss of generality focus on the design of optimal direct mechanisms. Any direct mechanism can be characterized by two functions: (1) $\pi(q, t) \in [0, 1]$ which is the probability of sending signal σ_1 to the buyer of type t at the state of nature q ; (2) $p(t) \geq 0$ which is the non-negative payment (due to Lemma 2) from the buyer. Note that, $[1 - \pi(q, t)]$ will be the seller's probability of sending signal σ_0 to the buyer of type t conditioned on state q .

Our main goal is to derive a feasible and optimal π^*, p^* for the seller's information pricing mechanism. Note that this is *functional optimization* problem since both $\pi(q, t), p(t)$ are function variables that depend on continuous variable $t \in [t_1, t_2] (= T)$ and abstract variable q in measurable set Q . We thus refer to π as the *signaling function* and p as the *payment function*. Next, we will formulate the problem based on the constraints described by the revelation principle in Lemma 1. We start by deriving Seller's revenue objective:

$$\text{Seller Objective: } \max_{\pi, p} \int_{t \in T} f(t)p(t) dt$$

The revelation principle shows that the signaling scheme will have two signals σ_1, σ_0 , each resulting in Buyer best response of action 1 and 0 respectively. This poses a constraint about the signaling function $\pi(q, t)$: (1) $\int_{q \in Q} \pi(q, t)v(q, t)g(q) dq \geq 0$ for any $t \in T$; (2) $\int_{q \in Q} [1 - \pi(q, t)]v(q, t)g(q) dq \leq 0$ for any $t \in T$. The first constraint above ensures that when signal σ_1 is recommended to the buyer of type t , the buyer's expected utility $\frac{E_{q \sim G}[\pi(q, t)v(q, t)]}{E_{q \sim G}[\pi(q, t)]}$ for the active action 1 should be at least 0, i.e., the utility of the passive action 0. Conversely, the second constraint ensures that conditioned on signal σ_0 , the expected buyer utility for action 1 $\frac{E_{q \sim G}[(1 - \pi(q, t))v(q, t)]}{E_{q \sim G}[1 - \pi(q, t)]}$ should be at most 0. These two constraints are also widely referred to as the *obedience* constraints in the literatures of information design [5, 6]. Slightly manipulating the second constraint above, we obtain $\int_{q \in Q} \pi(q, t)v(q, t)g(q) dq \geq \int_{q \in Q} v(q, t)g(q) dq = v(t)$. Therefore, we can conveniently summarize the obedience constraint as follows:

$$\text{Obedience: } \int_{q \in Q} \pi(q, t)v(q, t)g(q) dq \geq \max\{0, v(t)\}, \forall t \in T \quad (4)$$

Given the obedience constraint and signaling scheme for type t , a buyer of true type t should be incentivized to take action 1 whenever receiving signal σ_1 and action 0 otherwise. Therefore, we can also think of the signaling scheme as *obedient action recommendations*. Note that Buyer derives utility 0 from the recommended action 0 as well, therefore the expected utility of buyer type t when he reports his type *truthfully* and follows π 's obedient recommendation is

$$u(t) = \mathbf{E}_{q \sim G}[\pi(q, t)v(q, t)] - p(t) = \int_{q \in Q} \pi(q, t)v(q, t)g(q) dq - p(t) \quad (5)$$

where the first term comes from utility from his decision making under Seller's signaling scheme and the second term is the payment to the seller. To ensure that Buyer is willing to participate in the mechanism, we will impose the following individual rationality (IR) constraint:

$$\text{IR: } \int_{q \in Q} \pi(q, t)v(q, t)g(q) dq - p(t) \geq \max\{0, v(t)\}, \forall t \in T, \quad (6)$$

where the right-hand side is the buyer's expected utility of not participating in the mechanism. We remark that if the buyer does not participate in the information selling mechanism, he still has the freedom to pick the better action among 1, 0 based on his prior beliefs about q , leading to a utility equal to $\max\{0, v(t)\}$ under no information purchase. Therefore, we also define the surplus $s(t)$ – the additional utility gain of participating the mechanism – as a function of the buyer type t to be

$$s(t) = u(t) - \max\{v(t), 0\} = \int_{q \in Q} \pi(q, t)v(q, t)g(q) dq - p(t) - \max\{v(t), 0\}. \quad (7)$$

The IR constraint is equivalent to non-negative surplus. Interestingly, since the payment function is always non-negative, the IR Constraint 6 implies the Obedience Constraint (4).

Derivation of Incentive Compatibility (IC) Constraints. Finally, the IC turns out to require some careful derivations. To guarantee IC, we require that a buyer of type t should obtain higher utility than mis-reporting any type t' . This turns out to require some analyses since when a type- t buyer misreports type t' , the signaling scheme described by $\{\pi(q, t')\}_{q \in Q}$ may not be obedient any more. Therefore, the buyer utility for signal σ_1 should be the maximum between 0 and the following expected utility for action 1

$$U_1(t'; t) := \int_{q \in Q} \pi(q, t')v(q, t)g(q) dq = \int_{q \in Q} \pi(q, t')v_1(q)[t + \rho(q)]g(q) dq. \quad (8)$$

Conversely, the buyer utility for signal σ_0 will be the maximum between 0 and the following expected utility for action 1

$$\int_{q \in Q} [1 - \pi(q, t')] \times v_1(q)[t + \rho(q)]g(q) dq = v(t) - U_1(t'; t). \quad (9)$$

Consequently, the type- t buyer's utility from signaling scheme $\{\pi(q, t')\}_{q \in Q}$ equals $\max\{U_1(t'; t), 0\} + \max\{v(t) - U_1(t'; t), 0\}$. Therefore, the incentive compatibility constraint becomes the following complex constraint:

$$u(t) \geq \max\{U_1(t'; t), 0\} + \max\{v(t) - U_1(t'; t), 0\} - p(t') \quad (10)$$

Fortunately, it turns out that some cases of the above constraint can be implied by previous constraints. To see this, we distinguish between two cases:

1. When $t > t'$, we have $U_1(t'; t)$ defined in Equation (8) to be at least $U_1(t'; t')$, which is at least 0 by the Obedience Constraint (4) for t' . In this case, the right-hand-side of the above constraint becomes $U_1(t'; t) + \max\{v(t) - U_1(t'; t), 0\} - p(t')$, or equivalently $\max\{v(t), U_1(t'; t)\} - p(t')$. Note that $u(t) \geq v(t) - p(t')$ is already implied by the IR constraint $u(t) \geq v(t)$ and the condition $p(t') \geq 0$. Therefore, the only non-redundant constraint in this case is $u(t) \geq U_1(t'; t) - p(t')$.

2. When $t < t'$, we have $v(t) - U_1(t'; t)$ defined in Equation (9) to be at most $v(t') - U_1(t'; t')$, which is at most 0 by the Obedience Constraint (4) for t' . In this case, the right-hand-side of the above constraint becomes $\max\{U_1(t'; t), 0\} - p(t')$. Note that $u(t) \geq -p(t')$ is already implied by the IR constraint $u(t) \geq 0$ and the condition $p(t') \geq 0$. Therefore, the only non-redundant constraint in this case is also $u(t) \geq U_1(t'; t) - p(t')$.

Consequently, given the IR and Obedience constraints before, the IC constraint can finally be expressed as follows:

$$\text{IC: } \int_{q \in Q} \pi(q, t) v(q, t) g(q) dq - p(t) \geq \int_{q \in Q} \pi(q, t') v(q, t) g(q) dq - p(t'), \forall, t, t' \in T \quad (11)$$

Differences from Classic Mechanism Design for Goods. First, the mechanism in our setup has different design space and is characterized by a different set of variables. For example, there appears no natural correspondence between our the signaling scheme functional variable π and variables in classic mechanism design. The second important difference between selling information and classic mechanism design is that the IR constraint (6) in our setting is different from the IR constraint in classic mechanism design, which simply requires the utility of participation is at least 0. In our setting, however, a buyer of type t will have utility $\max\{0, v(t)\}$ without additional information. Therefore, our IR constraint has to guarantee that the buyer's utility from the mechanism is at least $\max\{0, v(t)\}$. This important difference turns out to significantly change the problem structure and raise new challenges to the design of the information selling mechanism. As will be shown later, it leads to very different optimal mechanisms from what we see in classic mechanism design.

3 The Optimal Mechanism

In this section, we present the characterization of the optimal pricing mechanism. Mathematically, we derive an optimal solution in closed form to the functional optimization problem formulated in Section 2. The optimal mechanism we obtain turns out to belong to the following category of *threshold mechanisms*.

Definition 1. [Threshold Mechanism] A mechanism (π, p) is called a threshold mechanism if there exists a function $\theta(t)$, such that for any $t \in [t_1, t_2]$,

$$\pi(q, t) = \begin{cases} 1 & \text{if } \rho(q) \geq \theta(t) \\ 0 & \text{otherwise} \end{cases}.$$

Since π is fully described by $\theta(t)$ here, they are both referred to as a threshold signaling function.

Note that the term ‘‘threshold’’ is only a property about the signaling function π and does not pose any constraint on the payment function p . To state our mechanism, we will need the following notions of *lower* and *upper* virtual values.

Definition 2 (Lower/Upper Virtual Values). For any type t with PDF $f(t)$ and CDF $F(t)$, the function $\underline{\phi}(t) = t - \frac{1-F(t)}{f(t)}$ is called the lower virtual value function and $\bar{\phi}(t) = t + \frac{F(t)}{f(t)}$ is called the upper virtual value function. A lower/upper virtual value function is regular if it is monotone non-decreasing in t .

The lower virtual value function $\underline{\phi}(t)$ is precisely the virtual value function commonly used in classic mechanism design [17]. However, the upper virtual value function is a new format, which to our knowledge does not appear in previous literature. The regularity definition is standard.

When a virtual value function is irregular, we will need to apply the so-called ‘‘ironing’’ trick to make it monotone non-decreasing in t . Myerson [17] developed a procedure for ironing the lower virtual value

function $\underline{\phi}(t)$. This procedure can be easily generalized to ironing any function about the buyer type t , specifically, also our upper virtual value function $\bar{\phi}(t)$ and mixtures of $\bar{\phi}(t), \underline{\phi}(t)$. We defer the formal description of this ironing procedure to Appendix A.1 and only introduce them as a definition here.

Definition 3 (Mixed Virtual Values and Ironing). *For any $c \in [0, 1]$, define $\phi_c(t) = c\underline{\phi}(t) + (1 - c)\bar{\phi}(t)$ as a mixed virtual value function. For any virtual value function $\phi(t)$ (upper or lower or mixed), let $\phi^+(t)$ denote the ironed version of $\phi(t)$ obtained via the standard ironing procedure of Myerson [17].*

If a virtual value function $\phi(t)$ is already monotone non-decreasing, it will remain the same after the ironing process, i.e., $\phi^+(t) = \phi(t), \forall t$. The following monotonicity property of the ironed mixed virtual value functions will be needed for proving our main results. Its proof is a bit technical and is deferred to Appendix A.2.

Lemma 3. [Monotonicity of Ironed Mixed Virtual Values] *Define $\phi_c(t) = c\underline{\phi}(t) + (1 - c)\bar{\phi}(t)$. Then we have for any $0 \leq c < c' \leq 1$, $\phi_c^+(t) \geq \phi_{c'}^+(t)$ for any t . Moreover, $\phi_1^+(t) = \underline{\phi}^+(t) < t < \bar{\phi}^+(t) = \phi_0^+(t), \forall t \in (t_1, t_2)$.*

We are now ready to state the characterization of the optimal mechanism after introducing the following two quantities

$$V_L = \max\{v(t_1), 0\} + \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(x)} g(q)v_1(q) dq dx, \quad (12)$$

$$V_H = \max\{v(t_1), 0\} + \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\bar{\phi}^+(x)} g(q)v_1(q) dq dx, \quad (13)$$

where $\bar{\phi}^+(x)/\underline{\phi}^+(x)$ are the ironed upper/lower virtual value functions. Note that Lemma 3 implies $-\underline{\phi}^+(x) \geq -\bar{\phi}^+(x)$ and consequently $V_L \leq V_H$ since $g(q)v_1(q)$ is always non-negative and V_L integrates over a smaller region on q .

Our main result is summarized in the following theorem.

Theorem 1. [Characterization of the optimal pricing mechanism for selling information]

1. If $v(t_2) \leq V_L$, the threshold mechanism with threshold signaling function $\theta^*(t) = -\underline{\phi}^+(t)$ and the following payment function represents an optimal mechanism:

$$p^*(t) = \int_{q \in Q} \pi^*(q, t)g(q)v(q, t) dq - \int_{t_1}^t \int_{q \in Q} \pi^*(q, x)g(q)v_1(q) dq dx.$$

where π^* is determined by $\theta^*(t)$ as in Definition 1. Moreover, $p^*(t)$ is monotone non-decreasing for $t \in [t_1, t_2]$.

2. If $v(t_2) \geq V_H$, the threshold mechanism with threshold signaling function $\theta^*(t) = -\bar{\phi}^+(t)$ and the following payment function represents an optimal mechanism:

$$p^*(t) = \int_{q \in Q} \pi^*(q, t)g(q)v(q, t) dq + \int_t^{t_2} \int_{q \in Q} \pi^*(q, x)g(q)v_1(q) dq dx - v(t_2),$$

where π^* is determined by $\theta^*(t)$ as in Definition 1. Moreover, $p^*(t)$ is monotone non-increasing for $t \in [t_1, t_2]$.

3. If $V_L < v(t_2) < V_H$, define $\phi_c(t) = c\underline{\phi}(t) + (1-c)\bar{\phi}(t)$ to be the mixed virtual value function, where $c \in (0, 1)$ is a constant that satisfies

$$\int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq dt = v(t_2),$$

where $\phi_c^+(t)$ is the ironed version of $\phi_c(t)$. Then the threshold mechanism with threshold signaling function $\theta^*(t) = -\phi_c^+(t)$ and the following payment function represents an optimal mechanism:

$$p^*(t) = \int_{q \in Q} \pi^*(q, t) g(q) v(q, t) dq - \int_{t_1}^t \int_{q \in Q} \pi^*(q, x) g(q) v_1(q) dq dx.$$

Moreover, $p^*(t)$ is monotone non-decreasing in t when $F(t) \leq c$ and monotone non-increasing when $F(t) > c$.

Let \bar{t} satisfy $v(\bar{t}) = 0$. Then in all cases above, the buyer surplus function $s(t)$ is monotone non-decreasing when $t \leq \bar{t}$ and monotone non-increasing when $t \geq \bar{t}$.

Note that in the optimal mechanism of Theorem 1, if the signaling schemes for two types t, t' are the same, then their payment must also be the same, i.e., $p^*(t) = p^*(t')$. This is a simple consequence of the IC constraint — if $p^*(t) > p^*(t')$, the buyer of type t would misreport t' , and vice versa.

Remark 1. In all three cases of Theorem 1, a threshold mechanism is optimal. However, the format of the optimal mechanism and payment properties depend on how $v(t_2)$ compares to V_L, V_H . Note that threshold mechanisms are ubiquitous in reality. In many inspections, examinations and recommendations, we often see some goods (or services) pass a test (or deserve a recommendation). These can be viewed as a threshold signaling scheme. What we pay for conducting these tests or receiving recommendations are precisely the required payment for receiving such information. From this perspective, Theorem 1 characterizes the optimal signaling threshold and payment for buyers drawn from a random population.

Remark 2. We briefly discuss the choice of the constant c in Case 3 of Theorem 1. As we will show later in our proof, $v(t_2) \leq V_H$ will imply $v(t_1) \leq 0$ for any feasible mechanism. Therefore, in Case 3, the V_L, V_H defined in Equation (12) and (13) only has the integral term. Therefore, the condition of Case 3 boils down to

$$\int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(x)} g(q) v_1(q) dq dx < v(t_2) < \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\bar{\phi}^+(x)} g(q) v_1(q) dq dx.$$

Since $\underline{\phi}^+(x) < \bar{\phi}^+(x)$ for any x , the choice of $c \in (0, 1)$ is trying to “interpolating” the two integral region $\{q : \rho(q) \geq -\bar{\phi}^+(x)\}$ and $\{q : \rho(q) \geq -\underline{\phi}^+(x)\}$. Since we assume that the distribution has no point mass, the following expression

$$\int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(x)} g(q) v_1(q) dq dx$$

is continuous in c .⁴ Lemma 3 implies that it is also monotone weakly decreasing in c . Therefore, we can binary search for the c that makes the value of this integral equal exactly $v(t_2)$. This also leads to a tractable algorithm for computing the c parameter.

We conclude this section by describing two examples and illustrate what the optimal mechanism would be like in concrete instances.

⁴This is the situation where the assumption that the distribution of ρ has no point masses is needed. Without this assumption, the threshold mechanism will need randomization for those q with $\rho(q) = \phi_c^+(t)$. See Appendix G for the refined characterization of the optimal mechanism for general ρ .

3.1 An Example of Regular Case 1

Consider $v(q, t) = qt - 2 = v_1(q)[t + \rho(q)]$ where $v_1(q) = q$ and $\rho(q) = \frac{-2}{q}$. Suppose $q \in Q = [0, 1]$ is uniformly distributed, i.e., $g(q) = 1$. Let $t \in T = [2, 3]$ also be uniformly distributed with $f(t) = 1$. Among others, this utility function captures online advertising where q is the probability an Internet user will purchase the product of an advertiser (the information buyer) and t is the advertiser's value for selling a product. The constant 2 is interpreted as the advertiser's payment for displaying his ads to an Internet user.

In this instance, $\underline{\phi}(t)$ is non-decreasing, thus, ironing doesn't change its value. We have $\underline{\phi}^+(t) = \underline{\phi}(t) = t - \frac{1-F(t)}{f(t)} = 2t - 3$. Note that $v(t) = \int_{q \in Q} g(q)v(q, t) dq = \int_0^1 (tq - 2) dq = \frac{t}{2} - 2$ for any $t \in [2, 3]$. Since V_L defined in Equation (13) is clearly non-negative, we have $v(t_2) = -0.5 < 0 \leq V_L$, so the instance falls into Case 1 of Theorem 1. This implies that an optimal mechanism can be specified by a threshold signaling scheme $\theta^*(t) = -\underline{\phi}^+(t) = 3 - 2t$. That is, for any buyer type t the mechanism will make obedient recommendation of the active action 1 when $\rho(q) \geq -\underline{\phi}^+(t)$, or concretely, when $q \geq \frac{2}{2t-3}$. Now there are two situations.

- When $t \leq 2.5$, the mechanism will recommend action 1 when $q \geq \frac{2}{2t-3} \geq 1$, which means the mechanism will never recommend action 1. Therefore, $\pi^*(t, q) = 0$ for all $q \in Q$ in this situation and the payment $p^* = 0$. Therefore, for these buyer types, the seller will sell no information to them and charges them 0 as well.
- When $t > 2.5$, the mechanism will recommend action 1 when $q \geq \frac{2}{2t-3}$, which is a threshold in $(0, 1)$ and decreases in t . In this situation, the payment function $p^*(t)$ can then be computed as follows

$$p^*(t) = \int_{\frac{2}{2t-3}}^1 [(qt - 2)] dq - \int_{2.5}^t \int_{\frac{2}{2x-3}}^1 q dq dx = -0.25 + \frac{4t - 9}{(2t - 3)^2}$$

For these buyer types, their utility from the mechanism will be

$$u(t) = \int_{\frac{2}{2t-3}}^1 [(qt - 2)] dq - p^*(t) = -1.75 + \frac{t}{2} + \frac{1}{2t - 3}$$

Since without information, the optimal buyer action will be action 0 due to $v(t) \leq 0$, $u(t)$ also equals the buyer surplus and can be verified to be monotone increasing.

Notably, to achieve optimal revenue, the optimal mechanism does not simply recommend the active action 1 whenever $v(q, t) \geq 0$. For example, when $t = 2.3$, the mechanism reveals no information (and asks for no charges) even $v(q, t) > 0$. Therefore, the revenue-optimal mechanism is generally not optimal for the welfare. Indeed, it sacrifices the welfare for the buyers with smaller types to extract more revenue from the buyers with higher types.

3.2 An Example of Irregular Case 2

Consider another instance $v(q, t) = 10q(t+q-72) = v_1(q)[t+\rho(q)]$ where $v_1(q) = 10q$ and $\rho(q) = q - 72$. Suppose $q \in Q = [0, 60]$ is uniformly distributed, i.e., $g(q) = \frac{1}{60}$. Let $t \in T = [30, 60]$ to be piece-wise uniformly distributed. Specifically, t be uniformly distributed in $[30, 54]$ with $f(t) = \frac{1}{48}$ and $[54, 60]$ with $f(t) = \frac{1}{12}$. This value function may capture house selling where t is the buyer's private preference on the house quality q , and $v_1(q)\rho(q)$ models the net value this house can bring independently from personal preferences.

$$\bar{\phi}(t) = \begin{cases} 2t - 30 & \text{if } 30 \leq t < 54 \\ 2t - 48 & \text{if } 54 \leq t \leq 60 \end{cases}$$

$$\bar{\phi}^+(t) = \begin{cases} 2t - 30 & \text{if } 30 \leq t < 48 \\ 66 & \text{if } 48 \leq t < 57 \\ 2t - 48 & \text{if } 57 \leq t \leq 60 \end{cases}$$

Algebraic calculation shows that the (irregular) upper virtual value and its ironed version can be listed above. We can verify that this instance falls into Case 2 of Theorem 1 because $v(t_2) = 8400 > \max(0, v(t_1)) + V_H = 7944$. Thus, Theorem 1 implies that an optimal mechanism can be specified by a threshold signaling scheme $\theta^*(t) = -\bar{\phi}^+(t)$. That is, for any buyer type t the mechanism will make obedient recommendation of the active action 1 when $\rho(q) \geq -\bar{\phi}^+(t)$. One can verify that in this case, the optimal mechanism will reveal a non-trivial amount of information to every type of buyers (except the only buyer type t_2) and also have positive charge from them. Concretely, the payment function $p^*(t)$ for this instance can be calculated as follows.

$$p(t) = \begin{cases} \frac{2(t-51)^3}{9} + 7(t-51)^2 - 27 & \text{if } 30 \leq t < 48 \\ 30 & \text{if } 48 \leq t < 57 \\ \frac{2(t-60)^3}{9} + 4(t-60)^2 & \text{if } 57 \leq t \leq 60 \end{cases}$$

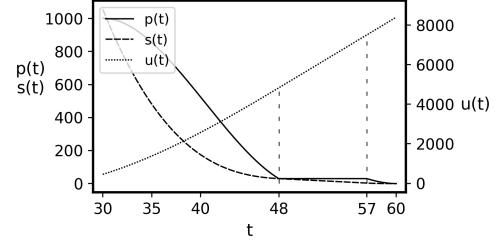


Figure 1: Functions in example 2

Figure 1 plots the (no-increasing) payment $p(t)$ as a function of buyer types t , as well as the buyer's utility $u(t)$ and surplus $s(t)$ function, for which we omit the standard calculations and solutions. Note that a large utility $u(t)$ does not mean a large surplus since the buyer with a large type may originally already has a very large utility, and participating the mechanism will not give them much additional utility, i.e., the surplus. This is exactly the situation illustrated in Figure 1.

One interesting observation happens at the interval $48 \leq t < 57$. The mechanism recommends action 1 when $(q - 72) \geq -66$, or equivalent $q \geq 6$, for all these t . Moreover, the mechanism also charges the same amount $p(t) = 30$ for t in this interval. This is predicted by the fact that for any two buyer types, the optimal mechanism charges the same payment from them if recommendation policies for them are the same. As we discussed earlier, this fact is induced by IC.

Another interesting observation is that to the optimal mechanism sometimes recommends the active action 1 even when $v(q, t) < 0$. For example, when $t = 40$, the mechanism recommends action 1 whenever $q \geq 22$. However, when $q = 22$, buyer's valuation is actually negative $v(q, t) = -2200$. Notably, in Case 2, the mechanism will always recommend action 1 whenever $v(q, t) \geq 0$ while in Case 1 this is not true. This is due to the nature of $\bar{\phi}^+(t) \geq t \geq \underline{\phi}^+(t)$.

Note that, the surplus $s(t)$ is the extra utility Buyer can get by participating in the mechanism and is defined in (7). Non-negativity of this function guarantees IR constraint. We shall show that it is monotone non-increasing in Case 2 (as plotted in Figure 1), monotone non-decreasing in Case 1, and will increase first then decrease in Case 3. The optimal mechanism will make $s(t_1) = 0$ in Case 1, $s(t_2) = 0$ in Case 2, and $s(t_1) = s(t_2) = 0$ in Case 3.

4 (Partial) Proof of Theorem 1

In this section, we prove Theorem 1. Due to space limit, we will only provide a proof for Case 3. The core idea for proving Case 1 and 2 are similar; we thus defer them to Appendix E and F, respectively. As we will see, our derivation here differs significantly from the derivation of optimal mechanisms for goods. The proof can be divided into two major steps: (1) characterizing the space of feasible mechanisms; (2) deriving the optimal mechanism within the feasible space. While the first step is also based on analysis of the IC constraints like that in classic mechanism design, the conclusions we obtain are quite different since our IC constraints are different. The second step deviates significantly from classic approaches, and is arguably much more involved due to additional constraints that we have to handle.

4.1 Characterization of Feasible Mechanisms

We first characterize the set of *feasible* mechanisms that satisfy the IR Constraints (6) and the IC Constraints (11). By Lemma 2, we can focus on the space of mechanisms with positive payments. In this case, the Obedience Constraints (4) can be implied by the IR constraints.

To describe our characterization, it is useful to introduce the following quantity.

$$P_\pi(t) = \int_{q \in Q} \pi(q, t) \cdot g(q) v_1(q) dq \quad (14)$$

Note that $P_\pi(t)$ can be interpreted as the expected *weighted* probability of being recommended the active action 1 where the weight is $v_1(q)$. The following lemma summarizes our characterization. To give readers some intuition, we only provide a proof of sufficiency here and defer the proof of necessity to Appendix C.

Lemma 4 (Characterization of Feasible Mechanisms). *A mechanism (π, p) with non-negative payments is feasible if and only if it satisfies the following constraints:*

$$P_\pi(t) \text{ is monotone non-decreasing in } t \quad (15)$$

$$u(t) = u(t_1) + \int_{t_1}^t P_\pi(x) dx, \forall t \in T \quad (16)$$

$$u(t_2) \geq v(t_2), \quad u(t_1) \geq 0 \quad (17)$$

$$p(t) \geq 0, \quad \forall t \in T \quad (18)$$

Proof of Sufficiency. We prove that Constraints (15)-(18) imply Obedience (4), IR (6) and IC (11) constraints. IC constraint (11) is equivalent to

$$u(t) \geq u(t') + \int_{q \in Q} \pi(q, t') \cdot g(q) [v(q, t) - v(q, t')] dq = u(t') + (t - t') P_\pi(t').$$

Therefore Constraints (15) and (16) imply IC constraint (11) because if $t' < t$, we have

$$u(t) - u(t') = \int_{t'}^t P_\pi(x) dx \geq \int_{t'}^t P_\pi(t') dx = (t - t') P_\pi(t').$$

Similarly, when $t' > t$, we also have $u(t) - u(t') \geq (t - t') P_\pi(t')$.

The IR constraint (6) is equivalent to $u(t) \geq 0$ and $u(t) \geq v(t)$. Since $P_\pi(x) \geq 0$, Constraint (16), together with $u(t_1) \geq 0$, imply $u(t) \geq 0$ for any t . We now leverage $u(t_2) \geq v(t_2)$ to prove $u(t) \geq v(t)$ for any t , as follows:

$$u(t) = u(t_1) + \int_{t_1}^t P_\pi(x) dx = u(t_2) - \int_t^{t_2} P_\pi(x) dx \geq v(t_2) - \int_t^{t_2} P_\pi(x) dx.$$

Using the definition of $v(t_2)$ and $P_\pi(x)$, we get

$$\begin{aligned} u(t) &= \int_{q \in Q} g(q) v_1(q) [t_2 + \rho(q)] dq - \int_t^{t_2} \int_{q \in Q} \pi(q, x) g(q) v_1(q) dq dx \\ &\geq \int_{q \in Q} g(q) v_1(q) [t_2 + \rho(q)] dq - \int_t^{t_2} \int_{q \in Q} g(q) v_1(q) dq dx \\ &= \int_{q \in Q} g(q) v_1(q) [t + \rho(q)] dq \\ &= v(t) \end{aligned}$$

Finally, the Obedience constraint (4) follows from IR constraint (6) and $p(t) \geq 0$. \square

Note that Condition (15) is analogous to Myerson's allocation monotonicity condition as in auction design, but is different. Specifically, in Myerson's optimal auction, the value of an item directly depends on buyer type t with no weight associated with it. In information selling, the value of taking the active action will depend on the utility coefficient $v_1(q)$.

Next we characterize Buyer's surplus from participating in the information selling mechanism, defined as follows:

$$\text{Buyer Surplus: } s(t) = u(t) - \max\{0, v(t)\}. \quad (19)$$

That is, the Buyer surplus is the difference between his utility from the information selling mechanism and his utility from directly picking the better action among 0, 1 without purchasing any information. Note that the IR constraint (6) is equivalent to $s(t) \geq 0$. Recall that Buyer of type t has expected utility $v(t) = \int_{q \in Q} v(q, t)g(q) dq$ for action 1 without purchasing any information. Since $v(q, t)$ is monotone non-decreasing in t , we know that $v(t)$ is also monotone non-decreasing. Let \bar{t} be the Buyer type at which $v(t)$ crosses 0 (\bar{t} can be any one if there are multiple such t). The following lemma characterize how Buyer's surplus changes as a function of his type.

Lemma 5. *Let \bar{t} be any buyer type such that $v(\bar{t}) = \int_{q \in Q} v(q, \bar{t})g(q) dq = 0$. In any feasible mechanism (π, p) with non-negative payments, the buyer's surplus $s(t)$ is non-negative for $t \in [t_1, t_2]$, monotone (weakly) increasing for $t \in [t_1, \bar{t}]$, and monotone (weakly) decreasing for $t \in [\bar{t}, t_2]$.*

Proof. When $t \leq \bar{t}$, we have $v(t) \leq 0$. Therefore, without participating in the mechanism to purchase additional information, Buyer will derive utility 0 by taking the passive action 0. So his surplus for participation is

$$s(t) = u(t) = u(t_1) + \int_{t_1}^t P_\pi(x) dx$$

by the utility identify in Equation (17). Since $u(t_1) \geq 0$ and $P_\pi(x) \geq 0$, it is easy to see that $s(t)$ is non-negative and monotone non-decreasing in t .

When $t \geq \bar{t}$, we have $v(t) \geq 0$. So Buyer will derive utility $v(t)$ without participating the information selling mechanism. We thus have

$$\begin{aligned} s(t) &= u(t) - v(t) \\ &= \left[u(t_1) + \int_{t_1}^t \int_{q \in Q} \pi(q, x)v_1(q)g(q) dq dx \right] - \left[\int_{q \in Q} v_1(q)[t + \rho(q)]g(q) dq \right] \\ &= \left[u(t_1) + \int_{t_1}^t \int_{q \in Q} \pi(q, x)v_1(q)g(q) dq dx \right] - \left[\int_{t_1}^t \int_{q \in Q} v_1(q)g(q) dq dx + v(t_1) \right] \\ &= u(t_1) - v(t_1) + \left[\int_{t_1}^t \int_{q \in Q} [\pi(q, x) - 1]v_1(q)g(q) dq dx \right]. \end{aligned}$$

Since $\pi(q, x) - 1 \leq 0$ and $v_1(q)g(q) \geq 0$, we thus have $s(t)$ is monotone non-increasing in t . Consequently, $s(t) \geq s(t_2) = u(t_2) - v(t_2) \geq 0$ by Inequality (17). \square

4.2 Deriving the Optimal Mechanism for Case 3

With the characterization of feasible mechanisms in Lemma 4, we are now ready to derive the optimal mechanism. This is where our proof starts to significantly deviate from typical approaches for classic mechanism design. To see the reasons, recall that Lemma 5 shows that Buyer surplus $s(t)$ in our problem will increase first and then decrease. In, e.g., feasible single-item mechanisms, Buyer's utilities are always increasing in their types, therefore the optimal auction should always set Buyer's surplus to be 0 at the lowest type. In our case, however, both $s(t_1)$ and $s(t_2)$ could be the lowest surplus and we have to figure out which one

needs to be the lowest and under what conditions. Second, to guarantee feasibility, the mechanism has to be designed to satisfy $u(t_2) \geq v(t_2)$ (coming from the IR constraints $u(t) \geq v(t), \forall t \in T$). This constraint is unique in selling information and is not required when designing mechanisms for selling goods. Guaranteeing feasibility of this constraint will result in much more intricate derivations as well as more intricate optimal mechanisms.

It turns out that whether the *minimum* buyer surplus will be at t_1 or at t_2 or simultaneously at both t_1, t_2 depends on how large $v(t_1)$ and $v(t_2)$ are. In fact, the optimal mechanism has different forms depending on whether $v(t_2) \leq V_L$, $v(t_2) \geq V_H$, or $V_L < v(t_2) < V_H$. We start with a technical lemma showing that the conditions for the three cases above can also be written in terms of $v(t_1)$ as well. The proof of this technical Lemma 6 is deferred to Appendix D.1.

Lemma 6. *Define*

$$V'_L = - \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(x)} g(q) v_1(q) dq dx \quad \text{and} \quad V'_H = - \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\bar{\phi}^+(x)} g(q) v_1(q) dq dx.$$

Then the three conditions $v(t_2) \leq V_L$, $v(t_2) \geq V_H$, and $V_L < v(t_2) < V_H$ are equivalent to $v(t_1) \leq V'_L$, $v(t_1) \geq V'_H$, and $V'_L < v(t_1) < V'_H$, respectively.

From now on, we will focus on the case with $V_L < v(t_2) < V_H$, and re-state the Case 3 of Theorem 1 in the following proposition.

Proposition 1. *If $V_L < v(t_2) < V_H$, define $\phi_c(t) = c\underline{\phi}(t) + (1 - c)\bar{\phi}(t)$ to be the mixed virtual value function, where $c \in (0, 1)$ is a constant that satisfies*

$$\int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq dt = v(t_2), \quad (20)$$

where $\phi_c^+(t)$ is the ironed version of $\phi_c(t)$. Then the threshold mechanism with threshold signaling function $\theta^(t) = -\phi_c^+(t)$ and the following payment function represents an optimal mechanism:*

$$p^*(t) = \int_{q \in Q} \pi^*(q, t) g(q) v(q, t) dq - \int_{t_1}^t \int_{q \in Q} \pi^*(q, x) g(q) v_1(q) dq dx.$$

Moreover, $p^*(t)$ is non-decreasing in t when $F(t) \leq c$ and monotone non-increasing when $F(t) > c$.

Before proving the optimality of our mechanism, we first argue that the constant c described in Proposition 1 actually exists.

Lemma 7. *If $V_L < v(t_2) < V_H$, there exists a constant $c \in (0, 1)$ that satisfies Equation (20).*

The proof of Lemma 7 is deferred to Appendix D.2. But we emphasize that the proof used the assumption that the distribution of $\rho(q)$ does not contain a point mass. However, even if this assumption does not hold, we can slightly modify our mechanism to get a threshold mechanism with partial recommendations on threshold boundary, and still achieve the optimal revenue. For clarity, we put the solution and proof for that general case in Appendix G.

Lemma 7 implies that the mechanism proposed in Proposition 1 exists. Now we show that it is also feasible.

Lemma 8. *The mechanism (π^*, p^*) defined according to $\phi_c^+(t)$ is feasible.*

Proof. To prove Lemma 8, it suffices to show that mechanism (π^*, p^*) satisfies all the constraints (15), (16), (17), and (18). By definition,

$$P_{\pi^*}(t) = \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq.$$

Since $\phi_c^+(t)$ is already ironed, it is non-increasing in t . Thus the integral domain of $P_{\pi^*}(t)$ gets larger as t increases. So $P_{\pi^*}(t)$ is non-decreasing since $g(q)v_1(q) \geq 0$, satisfying constraint (15).

To show that the mechanism satisfies constraint (16), note that

$$u(t) = \int_{q \in Q} g(q) \pi^*(q, t) v(q, t_1) dq - p(t) = \int_{t_1}^t P_{\pi^*}(x) dx.$$

Thus $u(t_1) = 0$ and

$$u(t) = u(t_1) + \int_{t_1}^t P_{\pi^*}(x) dx.$$

As for constraint (17), we already have $u(t_1) = 0$. And

$$u(t_2) = \int_{t_1}^{t_2} P_{\pi^*}(x) dx = \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq dt = v(t_2),$$

where the last equation is the definition of the constant c .

Finally, we show that the payment is non-negative, i.e., mechanism (π^*, p^*) satisfies constraint (18). We claim that $p^*(t)$ is monotone non-decreasing when $F(t) \leq c$, and monotone non-increasing when $F(t) \geq c$ (recall that $c \in (0, 1)$).

Define t_c as the buyer type where $F(t_c) = c$. Since $\phi_c(t) = c\underline{\phi}(t) + (1-c)\bar{\phi}(t) = t + \frac{F(t)-c}{f(t)}$, we have $\forall t, s.t. F(t) \leq c, \phi_c(t) \leq t$ and $\forall t, s.t. F(t) \geq c, \phi_c(t) \geq t$.

Following arguments similar to proof of Lemma 3, we know that $\phi_c^+(t) \leq t, \forall t \leq t_c$ and $\phi_c^+(t) \geq t, \forall t \geq t_c$. For any $t < t_c$, let t' be any number in the interval $[\phi_c^+(t), t]$. Thus $\phi_c^+(t) \geq \phi_c^+(t')$. And

$$\begin{aligned} p^*(t) - p^*(t') &= \int_{q \in Q} g(q) \pi^*(q, t) v(q, t) dq - \int_{q \in Q} g(q) \pi^*(q, t') v(q, t') dq - \int_{t'}^t P_{\pi^*}(x) dx \\ &= \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v(q, t) dq - \int_{q: \rho(q) \geq -\phi_c^+(t')} g(q) v(q, t') dq - \int_{t'}^t P_{\pi^*}(x) dx. \end{aligned}$$

When $\rho(q) \geq -\phi_c^+(t)$, we have $v(q, t') = v_1(q)[t' + \rho(q)] \geq v_1(q)[t' - \phi_c^+(t)] \geq 0$, where the last inequality is because of the choice of t' . So the second term in the above equation satisfies:

$$\begin{aligned} \int_{q: \rho(q) \geq -\phi_c^+(t')} g(q) v(q, t') dq &= \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v(q, t') dq - \int_{q: -\phi_c^+(t) \leq \rho(q) < -\phi_c^+(t')} g(q) v(q, t') dq \\ &\leq \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v(q, t') dq. \end{aligned}$$

Thus,

$$\begin{aligned} p^*(t) - p^*(t') &\geq \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v(q, t) dq - \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v(q, t') dq - \int_{t'}^t P_{\pi^*}(x) dx \\ &= \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q)(t - t') dq - \int_{t'}^t P_{\pi^*}(x) dx \\ &= (t - t') P_{\pi^*}(t) - \int_{t'}^t P_{\pi^*}(x) dx \\ &\geq 0, \end{aligned}$$

where the last inequality is due to the monotonicity of $P_{\pi^*}(t)$.

Therefore, the payment function $p^*(t)$ is monotone non-decreasing in the interval $[\underline{\phi}^+(t), t]$. Since the set of intervals $\{[\underline{\phi}^+(t), t] \mid t \in [t_1, t_c]\}$ covers $[t_1, t_c]$, we conclude that $p^*(t)$ is monotone non-decreasing in $[t_1, t_c]$.

Using similar analyses, we can show that $p^*(t)$ is monotone non-increasing in the interval $[t_c, t_2]$. Therefore, to prove that $p^*(t) \geq 0$ for all $t \in T$, it suffices to show that $p^*(t_1) \geq 0$ and $p^*(t_2) \geq 0$. Indeed, we have

$$p^*(t_1) = \int_{q \in Q} \pi^*(q, t_1) g(q) v(q, t_1) dq - u(t_1) = \int_{q: \rho(q) \geq -\phi_c^+(t_1)} g(q) v(q, t_1) dq \geq 0.$$

The inequality is because when $\rho(q) \geq -\phi_c^+(t_1) \geq -t_1$, we have $v(q, t_1) = v_1(q)[t_1 + \rho(q)] \geq 0$. And

$$\begin{aligned} p^*(t_2) &= \int_{q \in Q} \pi^*(q, t_2) g(q) v(q, t_2) dq - u(t_2) \\ &= \int_{\rho(q) \geq -\phi_c^+(t_2)} g(q) v_1(q)[t_2 + \rho(q)] dq - \int_{q \in Q} g(q) v_1(q)[t_2 + \rho(q)] dq \\ &= - \int_{\rho(q) < -\phi_c^+(t_2)} g(q) v_1(q)[t_2 + \rho(q)] dq \\ &\geq 0, \end{aligned}$$

where the inequality is because $\rho(q) < -\phi_c^+(t_2) \leq -t_2$. \square

To prove that the mechanism (π^*, p^*) is the optimal feasible mechanism, we need to apply the ironing trick. We will first derive the revenue function for any feasible mechanism using both t_1 and t_2 as reference points, and manipulate the expression so that it contains multiple terms. Then we show that the mechanism (π^*, p^*) optimizes all these terms simultaneously.

Now we are ready to show the proof of Proposition 1.

Proof of Proposition 1. Let (π, p) be any feasible mechanism. Since the utility of Buyer is just the difference between the value obtained from purchasing the item and their payment, we can write the revenue of Seller as:

$$REV(\pi, p) = \int_{t_1}^{t_2} f(t)p(t) dt = \int_{t_1}^{t_2} f(t) \left[\int_{q \in Q} g(q)\pi(q, t)v(q, t) dq - u(t) \right] dt.$$

Applying Equation (16), we get

$$\begin{aligned} REV(\pi, p) &= \int_{t_1}^{t_2} f(t) \left[\int_{q \in Q} g(q)\pi(q, t)v(q, t) dq - \int_{t_1}^t P_\pi(x) dx - u(t_1) \right] dt \\ &= \int_{t_1}^{t_2} f(t) \left[\int_{q \in Q} g(q)\pi(q, t)v(q, t) dq \right] dt - \int_{t_1}^{t_2} \int_{t_1}^t f(t)P_\pi(x) dx dt - u(t_1) \\ &= \int_{t_1}^{t_2} f(t) \left[\int_{q \in Q} g(q)\pi(q, t)v(q, t) dq \right] dt - \int_{t_1}^{t_2} \int_x^{t_2} f(t)P_\pi(x) dt dx - u(t_1) \\ &= \int_{t_1}^{t_2} f(t) \left[\int_{q \in Q} g(q)\pi(q, t)v(q, t) dq \right] dt - \int_{t_1}^{t_2} [1 - F(x)]P_\pi(x) dx - u(t_1), \end{aligned}$$

where the third equation comes from switching the order of integration. Thus

$$\begin{aligned}
REV(\pi, p) &= \int_{q \in Q} g(q) \left[\int_{t_1}^{t_2} f(t) \pi(q, t) v(q, t) dq \right] dt - \int_{t_1}^{t_2} [1 - F(t)] \int_{q \in Q} g(q) \pi(q, t) v_1(q) dq dt - u(t_1) \\
&= \int_{q \in Q} g(q) \left[\int_{t_1}^{t_2} f(t) \pi(q, t) \left(v(q, t) - v_1(q) \frac{1 - F(t)}{f(t)} \right) dt \right] dq - u(t_1) \\
&= \int_{q \in Q} g(q) \left[\int_{t_1}^{t_2} f(t) \pi(q, t) v_1(q) [\underline{\phi}(t) + \rho(q)] dt \right] dq - u(t_1).
\end{aligned} \tag{21}$$

The above derivation uses $u(t_1)$ as the “reference” points. Similarly, using a variant of Equation (16) $u(t) = u(t_2) - \int_t^{t_2} P_\pi(x) dx$, we can derive an alternative form of the revenue with $u(t_2)$ as the reference point:

$$REV(\pi, p) = \int_{q \in Q} g(q) \left[\int_{t_1}^{t_2} f(t) \pi^*(q, t) v_1(q) [\bar{\phi}(t) + \rho(q)] dt \right] dq - u(t_2). \tag{22}$$

Note that Equation (21) and (22) are just different representations of the (same) revenue of mechanism (π, p) . Thus any convex combination of them also represents the same revenue. Using the constant c given in the lemma as the combination coefficient, we have

$$\begin{aligned}
REV(\pi, p) &= c \left[\int_{q \in Q} g(q) \int_{t_1}^{t_2} f(t) \pi(q, t) v_1(q) [\underline{\phi}(t) + \rho(q)] dt dq - u(t_1) \right] \\
&\quad + (1 - c) \left[\int_{q \in Q} g(q) \int_{t_1}^{t_2} f(t) \pi(q, t) v_1(q) [\bar{\phi}(t) + \rho(q)] dt dq - u(t_2) \right] \\
&= \int_{t_1}^{t_2} \int_{q \in Q} [\phi_c(t) - \rho(q)] \pi(q, t) f(t) g(q) v_1(q) dq dt - cu(t_1) - (1 - c)u(t_2).
\end{aligned}$$

Let $H_c(\cdot), L_c(\cdot), h_c(\cdot), l_c(\cdot)$ be the corresponding functions when ironing the function $\phi_c(t)$ (detailed defined in Appendix A).

By definition, we have $h_c(F(t)) = \phi_c(t)$ and $l_c(F(t)) = \phi_c^+(t)$. So the first term in the right-hand side of the above equation can be written as

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{q \in Q} [\phi_c(t) + \rho(q)] \pi(q, t) f(t) g(q) v_1(q) dq dt \\
&= \int_{t_1}^{t_2} \int_{q \in Q} [\phi_c^+(t) + \rho(q)] \pi(q, t) f(t) g(q) v_1(q) dq dt \\
&\quad + \int_{t_1}^{t_2} \int_{q \in Q} [h_c(F(t)) - l_c(F(t))] \pi(q, t) f(t) g(q) v_1(q) dq dt.
\end{aligned}$$

Using integration by parts, we can simplify the second term as follows:

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{q \in Q} [h_c(F(t)) - l_c(F(t))] \pi(q, t) f(t) g(q) v_1(q) dq dt \\
&= \int_{t_1}^{t_2} [h_c(F(t)) - l_c(F(t))] P_\pi(t) dF(t) \\
&= [H_c(F(t)) - L_c(F(t))] P_\pi(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} [H_c(F(t)) - L_c(F(t))] dP_\pi(t)
\end{aligned}$$

Because L_c is the “convex hull” of H_c , so $L_c(0) = H_c(0)$ and $L_c(1) = H_c(1)$. Thus the first term above is simply 0. Therefore, we have

$$\begin{aligned} REV(\pi, p) &= \int_{t_1}^{t_2} \int_{q \in Q} [\phi_c^+(t) + \rho(q)] \pi(q, t) f(t) g(q) v_1(q) dq dt \\ &\quad - \int_{t_1}^{t_2} [H_c(F(t)) - L_c(F(t))] dP_\pi(t) - cu(t_1) - (1 - c)u(t_2). \end{aligned} \quad (23)$$

Now consider mechanism (π^*, p^*) , which is feasible according to Lemma 8. This mechanism clearly maximizes the first term in Equation (23) as $\pi^*(q, t) = 1$ if and only if $\phi_c^+(t) + \rho(q) \geq 0$. We also have $u(t_1) = 0$ and $u(t_2) = v(t_2)$ for this mechanism as shown in the proof of Lemma 8. Constraint (17) requires $u(t_1) \geq 0$ and $u(t_2) \geq v(t_2)$, which implies that this mechanism also optimizes the last two terms. As for the second term, note that $H_c(F(t)) - L_c(F(t)) \geq 0$ by definition, and $dP_\pi(t) \geq 0$ for any feasible mechanism. Thus the second term is always non-negative. However, we claim that with mechanism (π^*, p^*) , this term achieves 0. Clearly, the only interesting case is when $H_c(F(t)) - L_c(F(t)) > 0$. We will show that $dP_{\pi^*}(t) = 0$ for this case. In this case, t must lie in an ironed interval I . Thus $L_c(z)$ is linear in the interval I , where $z = F(t)$. This implies that $\phi_c^+(t) = l_c(z) = L'_c(z)$ is constant. So

$$P_{\pi^*}(t) = \int_{q \in Q} \pi^*(q, t) g(q) v_1(q) dq = \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq$$

is also constant in the interval I , which leads to $dP_{\pi^*}(t)$ being 0.

Therefore, the mechanism (π^*, p^*) optimizes all the 4 terms in Equation (23) simultaneously, thus is an optimal feasible mechanism. \square

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APPENDIX

A Ironing

A.1 Formal Description of the Ironing Procedure

Definition 4 (Ironing [17]). *Let t be the buyer's type with CDF $F(t)$ and PDF $f(t)$, and $\phi(t)$ be any function of the type t , called a virtual value function. The ironed function $\phi^+(t)$ can be obtained through the following process:*

1. Let $z = F(t)$ be another random variable and define $h(z) = \phi(F^{-1}(t))$, where $F^{-1}(t)$ is the inverse function of $F(t)$.

2. Define $H : [0, 1] \mapsto \mathbb{R}$ to be the integral of $h(z)$:

$$H(z) = \int_0^z h(r) \, dr.$$

3. Define $L : [0, 1] \mapsto \mathbb{R}$ be the “convex hull” of function H :

$$L(z) = \min_{z_1, z_2, \gamma} \{\gamma H(z_1) + (1 - \gamma)H(z_2)\},$$

where $z_1, z_2, \gamma \in [0, 1]$ and $\gamma z_1 + (1 - \gamma)z_2 = z$.

4. Let $l(z)$ be the derivative of L :

$$l(z) = L'(z).$$

5. Obtain $\phi^+(t)$ by variable substitution:

$$\phi^+(t) = l(z) = l(F(t)).$$

The above ironing trick is widely used in the literature. The process is illustrated in Figure 2. In Myerson's original work [17], he only considered ironing for the lower virtual value function $\underline{\phi}(t) = t - \frac{1-F(t)}{f(t)}$. However, this procedure generalizes to any virtual value function.

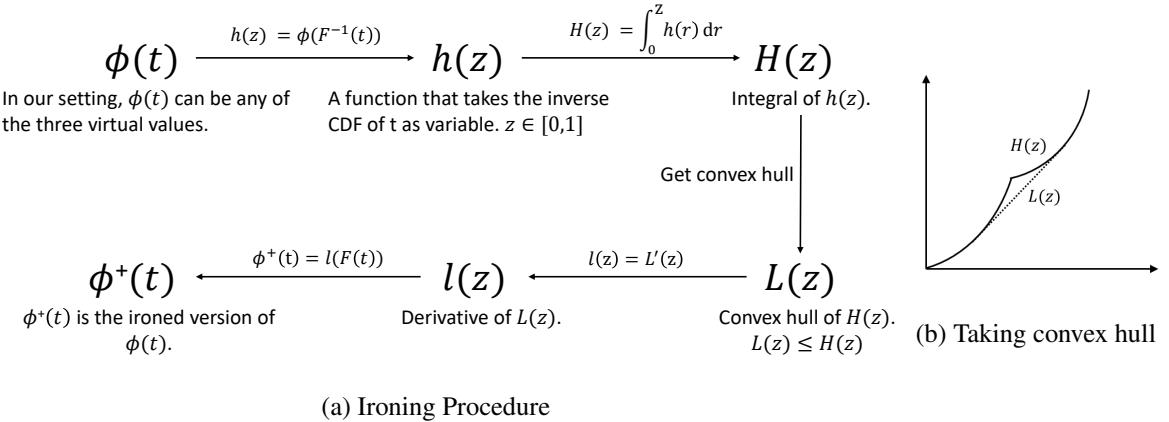


Figure 2: The ironing process

A.2 Proof of Lemma 3: Monotonicity of Mixed Virtual Values

Lemma (Restatement of Lemma 3). *Define mixed virtual value $\phi_c(t) = c\phi(t) + (1-c)\bar{\phi}(t)$. Then we have for any $0 \leq c < c' \leq 1$, $\phi_c^+(t) \geq \phi_{c'}^+(t)$ for any t . Moreover, $\phi_1^+(t) = \underline{\phi}^+(t) < t < \bar{\phi}^+(t) = \phi_0^+(t), \forall t \in (t_1, t_2)$.*

Proof. We first prove the monotonicity of $\phi_c^+(t)$ in c . We will instead consider the function ϕ in a new variable space $z = F(t) \in [0, 1]$, as opposed to the original space t . Therefore, let $l_c(z) = \phi_c^+(F^{-1}(z))$ and $h_c(z) = \phi_c(F^{-1}(z))$. Recall that the ironing procedure satisfies $L_c(z) = \int_0^z l_c(r) dr$ is the convex hull of $H_c(z) = \int_0^z h_c(r) dr$.

Note that during the ironing of function $H_c(z)$, it divided the compact variable space $[0, 1]$ into a finite number of small intervals with breaking points $0 = z_0, z_1, \dots, z_k = 1$ on which for any interval $[z_i, z_{i+1}]$: (1) either $H_c(z) = L_c(z)$ for any $z \in [z_i, z_{i+1}]$; (2) $l_c(z)$ is a constant and $H_c(z) \geq L_c(z)$ for any $z \in [z_i, z_{i+1}]$. In later case, we will call $[z_i, z_{i+1}]$ an *ironing interval* and say $H_c(z)$ is at *ironing state* in this interval. We call z_i the *ironing starting point* and z_{i+1} the *ironing ending point*. Similarly, in former case, we call $[z_i, z_{i+1}]$ an *un-ironing interval* and say $H_c(z)$ is at *un-ironing state* in this interval. Note that in this case, z_i will be an ironing ending point and z_{i+1} will be an ironing starting point. In fact, in the sequence $0 = z_0, z_1, \dots, z_k = 1$, ironing starting and ending points show up alternately. The following are a few useful properties that will be needed.

1. If $z \neq 0, 1$ is an ironing starting or ending point, then $h_c(z) = l_c(z)$ and $H_c(z) = L_c(z)$.
2. if $[z_i, z_{i+1}]$ is an ironing interval for $H_c(z)$, then we have $l_c(z) = \frac{H_c(z_{i+1}) - H_c(z_i)}{z_{i+1} - z_i}$ for any $z \in [z_i, z_{i+1}]$
3. For any $z \in [0, 1]$ we have $h_c(z) = t + \frac{F(t)}{f(t)} - \frac{c}{f(t)} \geq t + \frac{F(t)}{f(t)} - \frac{c'}{f(t)} = h_{c'}(z)$ where $t = F^{-1}(z)$.
4. Due to property (3) above, for any z we have $H_c(\bar{z}) - H_c(z) \geq H_{c'}(\bar{z}) - H_{c'}(z)$ for any $\bar{z} > z$. Moreover, $H_c(0) = H_{c'}(0) = 0$.

Similarly, we can also have a sequence of ironing starting and ending points for the function $H_{c'}(z)$. Let us merge all the ironing starting and ending points of $H_c(z)$ and $H_{c'}(z)$ together, and list them in order as $0 = z_0, z_1, \dots, z_k = 1$. Notably, within any interval $[z_i, z_{i+1}]$, both function $H_c(z)$ and $H_{c'}(z)$ can only have a single state, either ironing state or un-ironing state.

We first prove $l_c(0) \geq l_{c'}(0)$. This follows a case analysis about whether 0 is an ironing ending or starting point for H_c .

- If 0 is an ironing ending point for both $H_c, H_{c'}$, meaning both functions are not on ironing state at 0 and its neighborhood, we know $l_c(0) = h_c(0) \geq h_{c'}(0) = l_{c'}(0)$, as desired.
- If 0 is an ironing ending point only for H_c but an ironing starting point for $H_{c'}$, this means H_c is on un-ironing state at 0 and its neighborhood whereas $H_{c'}$ is on ironing state. Then we have we know $l_c(0) = h_c(0) \geq h_{c'}(0) \geq l_{c'}(0)$, as desired.
- If 0 is an ironing starting point for H_c (it does not matter it is an ironing ending or starting point for $H_{c'}$), this means H_c is on ironing state at 0 and its neighborhood. Let $\bar{z} \geq z_1$ be the immediate next ironing ending point for H_c . Suppose, for the sake of contradiction, that $l_c(0) < l_{c'}(0)$. We thus have

$$\begin{aligned}
H_c(\bar{z}) - H_c(0) &= l_c(0) \cdot (\bar{z} - 0) && \text{by Property (2) above} \\
&< l_{c'}(0) \cdot (\bar{z} - 0) && \text{by assumption} \\
&\leq L_{c'}(\bar{z}) - L_{c'}(0) && \text{by convexity of } L_{c'} \\
&\leq H_{c'}(\bar{z}) - H_{c'}(0) && L_{c'}(0) = H_{c'}(0), L_{c'}(\bar{z}) \leq H_{c'}(\bar{z})
\end{aligned}$$

This contradicts Property (4) above. Therefore, we must $l_c(0) \geq l_{c'}(0)$, as desired.

Next, we will prove that for any $i = 0, \dots, k - 1$ and any interval $[z_i, z_{i+1}]$ — more conveniently denoted as $[a, b]$ with interval lower bound a and upper bound b — we will have $l_c(z) \geq l_{c'}(z)$ for any $z \in [a, b]$.

Our proof uses an induction argument over the intervals indexed by i . Specifically, suppose we already have $l_c(a) \geq l_{c'}(a)$, we will show $l_c(z) \geq l_{c'}(z)$ for any $z \in [a, b]$. This, together with the base case for $a = 0$ as proved above, will prove the monotonicity of $l_c(z)$ on c .

The proof uses a case analysis about whether the ending point b of the interval $[a, b]$ is an ironing starting point or ending point for H_c or for $H_{c'}$. Note that there will be four cases here. This is because we do not know whether b is an ironing point on H_c or on $H_{c'}$ and thus have to consider both possibilities. Here, we will use the crucial property that both H_c and $H_{c'}$ will have the same state, i.e., ironing or un-ironing state, within $[a, b]$ due to our choice of a, b .

- If b is an ironing ending point for function $H_{c'}$, we have for any $z \in [a, b]$

$$\begin{aligned} l_c(z) &\geq l_c(a) && \text{by convexity of } L_c \\ &\geq l_{c'}(a) && \text{by induction hypothesis} \\ &= l_{c'}(z) && H_{c'} \text{ is at ironing state in } [a, b] \end{aligned}$$

- If b is an ironing starting point for function $H_{c'}$, this means $H_{c'}$ is at un-ironing state within $[a, b]$. If H_c is also at un-ironing state within $[a, b]$, then we have $l_c(z) = h_c(z) \geq h_{c'}(z) = l_{c'}(z)$ as desired. Thus, we now consider the case that H_c is at ironing state within $[a, b]$. Let $\bar{z} \geq b$ be the immediate next ironing ending point for H_c . Suppose, for the sake of contradiction, that $l_c(z) < l_{c'}(z)$ for some $z \in [a, b]$. Since H_c is at ironing state within $[a, \bar{z}]$, we know that $l_c(r) = l_c(z) < l_{c'}(z) \leq l_{c'}(r)$ for any $r \in [b, \bar{z}]$ since $l_{c'}(z)$ is monotone non-decreasing in z . Therefore, we have

$$\begin{aligned} H_c(\bar{z}) - H_c(b) &\leq L_c(\bar{z}) - L_c(b) && L_c(\bar{z}) = H_c(\bar{z}), L_c(b) \leq H_c(b) \\ &\leq l_c(b) \cdot (\bar{z} - b) && H_c \text{ is at ironing state in } [a, \bar{z}] \\ &< l_{c'}(b) \cdot (\bar{z} - b) && \text{by assumption} \\ &\leq L_{c'}(\bar{z}) - L_{c'}(b) && \text{by convexity of } L_{c'} \\ &\leq H_{c'}(\bar{z}) - H_{c'}(b) && b \text{ is an ironing starting point for } H_{c'} \end{aligned}$$

This contradicts Property (4) above. Therefore, we must $l_c(z) \geq l_{c'}(z)$ for all $z \in [a, b]$, as desired. Note that one corner case for this situation is when \bar{z} happens to equal b , i.e., b is both the ironing starting point of $H_{c'}$ and ironing ending point of H_c . Our argument above does not apply to this corner situation since the strict “ $<$ ” above becomes “ $=$ ”. However, this corner case can be proved via a simpler argument: $\forall z \in [a, b], l_c(z) = l_c(b) = h_c(b) \geq h_{c'}(b) \geq h_{c'}(z) = l_{c'}(z)$ where the second equality is due to the fact that b is an ironing ending point of H_c and the last equality is due to the fact that $H_{c'}$ is at un-ironing state within $[a, b]$.

- If b is an ironing starting point for function H_c , this means H_c is at un-ironing state within $[a, b]$. If $H_{c'}$ is also at un-ironing state within $[a, b]$, then we have $l_c(z) = h_c(z) \geq h_{c'}(z) = l_{c'}(z)$ as desired. If $H_{c'}$ is at ironing state within $[a, b]$, then we have $l_c(z) \geq l_c(a) \geq l_{c'}(a) = l_{c'}(z)$ as desired.
- Finally, if b is an ironing ending point for function H_c , this means H_c is at ironing state within $[a, b]$. If $H_{c'}$ is also at ironing state within $[a, b]$, then we have $l_c(z) = l_c(a) \geq l_{c'}(a) = l_{c'}(z)$ as desired. If $H_{c'}$ is at the un-ironing state within $[a, b]$, then we have $l_c(z) = l_c(b) = h_c(b) \geq h_{c'}(b) \geq l_{c'}(z)$ where: (1) the first equality is because H_c is at ironing state within $[a, b]$; (2) second equality is because b is an ironing ending point for H_c and the last inequality is because $H_{c'}$ is at un-ironing state within $[a, b]$ and thus $h_{c'}(z) = l_{c'}(z)$ is monotone non-decreasing in z .

Finally, we prove $\phi_1^+(t) = \underline{\phi}^+(t) < t < \bar{\phi}^+(t) = \phi_0^+(t), \forall t \in (t_1, t_2)$. We only show $\underline{\phi}^+(t) \leq t, \forall t$, as the other part follows from similar arguments.

Let \underline{H} and \underline{L} be the corresponding functions defined in Definition 4 when ironing the lower virtual value function $\phi(t)$. Define an ironed interval $I = (a, b) \subset [0, 1]$ to be an interval with $\underline{H}(a) = \underline{L}(a)$ and $\underline{H}(b) = \underline{L}(b)$ but $\underline{H}(z) > \underline{L}(z)$ for all $z \in I$. Thus the interval $[0, 1]$ can be partitioned into a set of disjoint ironed and non-ironed intervals.

For any t , if the corresponding $z = F(t)$ falls into a non-ironed interval, then we have $\underline{H}(z^*) = \underline{L}(z^*)$ for all z^* in the same interval. So $\underline{\phi}^+(t) = \underline{L}'(z) = \underline{H}'(z) = \underline{\phi}(t)$. Since $\underline{\phi}(t) \leq t$, we have $\underline{\phi}^+(t) \leq t$.

Therefore the only interesting case is when the corresponding $z = F(t)$ falls into an ironed interval $I = (a, b)$. In this case, function \underline{L} is linear in I , and $\underline{\phi}^+(z) = \underline{L}'(z) = \underline{L}'(a) = \underline{\phi}^+(a)$. But $\underline{H}'(a) > \underline{L}'(a)$ since I is an ironed interval and $\underline{H}(a+\epsilon) > \underline{L}(a+\epsilon)$ for any arbitrarily small ϵ . Therefore, $\underline{\phi}^+(t) = \underline{L}'(z) = \underline{L}'(a) < \underline{H}'(a) = \underline{\phi}(F^{-1}(a)) \leq F^{-1}(a) < F^{-1}(z) = t$. Thus, we can conclude $\underline{\phi}^+(t) \leq t, \forall t$. \square

B Proof of Technical Lemma 2

Lemma (Non-Negative Payments). *[Restatement of Lemma 2] There exists an optimal IC and IR mechanism in which $p_t \geq 0$ for all $t \in T$.*

Proof. Let (π, p) be any IC and IR optimal mechanism. We construct a different mechanism (π^*, p^*) which satisfies the same constraints and remains optimal but $p_t^* \geq 0$ for any t . For convenience, we divide buyer types into two sets: $T^+ = \{t \in T : p_t \geq 0\}$ is the set of types who have non-negative payments in mechanism (π, p) and $T^- = T \setminus T^+$ is the set of types who have negative payments.

The (π^*, p^*) is constructed from (π, p) as follows:

1. The mechanism for any $t \in T^+$ remains the same: for any $t \in T^+$, let $p_t^* = p_t$ and $\pi_t^* = \pi_t$ for all $q \in Q$;
2. The mechanism for any $t \in T^-$ becomes no information and no payment: for any $t \in T^-$, let $p_t^* = 0$, and π_t^* be the mechanism that reveals no information (e.g., sending a single signal).

We observe that the constructed mechanism (π^*, p^*) has three useful properties: (1) it yields revenue at least that of (π, p) by construction; (2) all buyer types' payments are non-negative now; (3) individual rationality constraint is satisfied for every buyer type. The third property follows from the construction: the utility of any buyer type $t \in T^+$ did not change and the utility of a type $t \in T^-$ now pays 0 and receives no information, so IR constraint is always satisfied.

However, the major issue with the constructed mechanism (π^*, p^*) is that it may not be incentive compatible, i.e., bidder type t may want to misreport t' . We first observe that the IC constraint for any $t \in T^+$ remains satisfied. First of all, any type $t \in T^+$ would not have incentive to deviate to another type $t' \in T^+$ due to the original IC constraint of (π, p) and the fact that the mechanism for types in T^+ remains the same. We claim that any type $t \in T^+$ would not have incentive to deviate to a type $t' \in T^-$ as well. This is because the information for $t' \in T^-$ is less (since Seller reveals no information now) and the payment is more (since $p_{t'}^* = 0 > p_{t'}$). Therefore, if in mechanism (π, p) buyer type t does not have incentives to deviate to t' , it remains to be true for (π^*, p^*) .

However, buyer type $t \in T^-$ may indeed have incentive to deviate to some type $t' \in T^+$ now, since they may want to receive beneficial information under some amount of payment. Here comes our last step of the construction — adjusting the above (π^*, p^*) to make any type $t \in T^-$ to also satisfy IC without decreasing the revenue neither violating the IR and obedient constraint. To do so, for any $t \in T^-$, let $t' \in T^+$ be the most profitable deviation of type t , i.e., the deviation that maximizes type t 's utility. We adjust (π^*, p^*) simply by adopting the scheme of type t' to the type t — i.e., resetting $\pi_t^* = \pi_{t'}$ and $p_t^* = p_{t'}$. After such adjustment, the IC constraint for any type $t \in T^-$ is satisfied by construction because each of these types

has indeed their most profitable mechanism. Meanwhile, this will also maintain the IC constraint for any type $t \in T^+$ since the adjustment did not add more menus. Note that IR constraint remains satisfied since the utility of any type $t \in T^+$ is non-decreasing in his adjustment. The revenue did not decrease as the payment $p^*(t)$ did not decrease in our adjustment for any $t \in T^+$. The only non-obvious part to verify is the obedience constraint. Indeed, the obedience constraint may be violated for type $t \in T^-$ during this adjustment since the recommended optimal action for the $t' \in T^+$ might not be optimal for t . To achieve obedience, we simply “rename” the recommended action for t to indeed be his optimal action. This restores the obedience constraint for t . Note that, this will either not change the revealed information or lead to less revealed information (when type t ’s optimal actions are the same under $\pi(\cdot, t')$), and thus will not hurt the IC constraints. \square

C Characterization of Feasible Mechanisms — Proof of Lemma 4

In this appendix section, we show that the conditions in Lemma 4 are also necessary for any feasible mechanism. We start by analyzing the IC Constraints. First, Constraint (11) can be re-arranged as follows:

$$\int_{q \in Q} [\pi(q, t) - \pi(q, t')] \cdot g(q)v(q, t) dq \geq p(t) - p(t').$$

Therefore, the IC constraint implies the following two inequalities about any two types t, t' :

$$\int_{q \in Q} [\pi(q, t) - \pi(q, t')] \cdot g(q)v(q, t) dq \geq p(t) - p(t'), \quad (24)$$

$$\int_{q \in Q} [\pi(q, t') - \pi(q, t)] \cdot g(q)v(q, t') dq \geq p(t') - p(t). \quad (25)$$

Combining Inequality (24) and (25), we obtain the following constraint for any pair of types t, t' :

$$\int_{q \in Q} [\pi(q, t') - \pi(q, t)] \cdot g(q)v(q, t) dq \leq p(t') - p(t) \leq \int_{q \in Q} [\pi(q, t') - \pi(q, t)] \cdot g(q)v(q, t') dq.$$

Therefore, the right-hand side of the above inequality must be at least its left-hand side. This implies the following *necessary condition* for any IC information selling mechanism (π, p) . That is, for any $t, t' \in T$, we have

$$\begin{aligned} 0 &\leq \int_{q \in Q} [\pi(q, t') - \pi(q, t)] \cdot g(q)[v(q, t') - v(q, t)] dq \\ &= [t' - t] \int_{q \in Q} [\pi(q, t') - \pi(q, t)] \cdot g(q)v_1(q) dq \end{aligned} \quad (26)$$

Recall the definition of $P_\pi(t)$ (14)

$$P_\pi(t) = \int_{q \in Q} \pi(q, t) \cdot g(q)v_1(q) dq.$$

Note that $P_\pi(t)$ can be interpreted as the expected *weighted* probability of being recommended the active action 1 where the weights are $v_1(q)$. A simple case analysis for $t' > t$ and $t' < t$ implies that Inequality (26) is equivalent to that $P_\pi(t)$ is monotone non-decreasing in t . We thus term this the *signaling monotonicity*. This is analogous to Myerson’s allocation monotonicity condition as in auction design, but is different. Specifically, in Myerson’s optimal auction, the value of an item directly depends on buyer type t with no weight associated with it. In information selling, the value of taking the active action will depend on the utility coefficient $v_1(q)$.

We now derive a relation between signaling scheme π and payment rule p for any IC mechanism. We start by analyzing Buyer's utility. Note that any buyer of type t will derive non-zero utilities only from the active action recommendation (i.e., action 1) since the passive action always leads to buyer utility 0. Therefore, as we defined in (5), Buyer of type t has the following utility:

$$\text{Utility of Buyer Type } t : \quad u(t) = \int_{q \in Q} [g(q)\pi(q, t)v(q, t)] \, dq - p(t)$$

Re-arranging Inequality (24), we have

$$\begin{aligned} u(t) &= \int_{q \in Q} [g(q)\pi(q, t)v(q, t)] \, dq - p(t) \\ &\geq \int_{q \in Q} [g(q)\pi(q, t')v(q, t)] \, dq - p(t') && \text{by Ineq. (24)} \\ &= \int_{q \in Q} [g(q)\pi(q, t')v(q, t)] \, dq + u(t') - \int_{q \in Q} [g(q)\pi(q, t')v(q, t')] \, dq \\ &= \int_{q \in Q} [g(q)\pi(q, t')[v(q, t) - v(q, t')]] \, dq + u(t') && \text{Algebraic Manipulation} \\ &= (t - t')P_\pi(t') + u(t') && \text{Def. of } P_\pi(t) \text{ and } v(q, t) \end{aligned}$$

As a result, Inequality (24) implies $u(t) - u(t') \geq (t - t')P_\pi(t')$. Together with a similar derivation from Inequality (25), we have the following inequality

$$(t - t')P_\pi(t') \leq u(t) - u(t') \leq (t - t')P_\pi(t).$$

Note that the above inequality holds for any t, t' . Therefore, by letting $t' \rightarrow t$ and invoking that fact that $P(t)$ is monotone and continuous, we can integrate the above equation from t_1 to t and obtain the inequalities:

$$\int_{t_1}^t P_\pi(x) \, dx \leq u(t) - u(t_1) \leq \int_{t_1}^t P_\pi(x) \, dx$$

This implies the following utility identify

$$u(t) = u(t_1) + \int_{t_1}^t P_\pi(x) \, dx.$$

Note that both the signaling monotonicity and the utility identify above are the necessary outcomes of the incentive compatibility constraints, more precisely, the outcome of Constraints (24) and (25).

D Omitted Proofs in Section 4.2

D.1 Proof of Technical Lemma 6

Lemma (Restatement of Lemma 6). *Define*

$$V'_L = - \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(x)} g(q)v_1(q) \, dq \, dx \quad \text{and} \quad V'_H = - \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\overline{\phi}^+(x)} g(q)v_1(q) \, dq \, dx.$$

Then the three conditions $v(t_2) \leq V_L$, $v(t_2) \geq V_H$, and $V_L < v(t_2) < V_H$ are equivalent to $v(t_1) \leq V'_L$, $v(t_1) \geq V'_H$, and $V'_L < v(t_1) < V'_H$, respectively.

Proof. We will only show that $v(t_2) \leq V_L$ is equivalent to $v(t_1) \leq V'_L$, as the other two cases follows from similar arguments.

By definition, we have

$$v(t_2) = \int_{q \in Q} g(q)v_1(q)[t_2 + \rho(q)] \, dx = v(t_1) + (t_2 - t_1) \int_{q \in Q} g(q)v_1(q) \, dq.$$

Thus $v(t_2) \leq V_L$ can be written as:

$$v(t_1) + (t_2 - t_1) \int_{q \in Q} g(q)v_1(q) \, dq \leq \max\{v(t_1), 0\} + \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(x)} g(q)v_1(q) \, dq \, dx.$$

Some re-arrangements yields:

$$v(t_1) - \max\{v(t_1), 0\} \leq \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(x)} g(q)v_1(q) \, dq \, dx - (t_2 - t_1) \int_{q \in Q} g(q)v_1(q) \, dq,$$

which is equivalent to:

$$\min\{v(t_1), 0\} \leq - \int_{t_1}^{t_2} \int_{q: \rho(q) \leq -\underline{\phi}^+(x)} g(q)v_1(q) \, dq \, dx = V'_L.$$

Note that the right-hand side is always non-positive. So the left-hand side has to be $v(t_1)$. Thus the condition $v(t_2) \leq V_L$ is equivalent to $v(t_1) \leq V'_L$, and also implies that $v(t_1) \leq 0$. \square

D.2 Proof of Technical Lemma 7

Lemma (Restatement of Lemma 7). *If $V_L < v(t_2) < V_H$, there exists a constant $c \in (0, 1)$ that satisfies Equation (20).*

Proof. Lemma 6 implies that the condition $v(t_2) < V_H$ is equivalent to the following:

$$v(t_1) < - \int_{t_1}^{t_2} \int_{q: \rho(q) \leq -\bar{\phi}^+(t)} g(q)v_1(q) \, dq \, dt. \quad (27)$$

The right-hand side of the above inequality is clearly non-positive. Thus $v(t_1) \leq 0$ and $\max\{v(t_1), 0\} = 0$. The condition $V_L < v(t_2) < V_H$ can be written as:

$$\int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q)v_1(q) \, dq \, dt < v(t_2) < \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q)v_1(q) \, dq \, dt.$$

When $c = 0$, we have $-\underline{\phi}^+(t) = -\phi_c^+(t)$ and

$$\int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q)v_1(q) \, dq \, dt = \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q)v_1(q) \, dq \, dt < v(t_2).$$

When $c = 1$, we have $-\bar{\phi}^+(t) = -\phi_c^+(t)$ and

$$\int_{t_1}^{t_2} \int_{\rho(q) \geq -\phi_c^+(t)} g(q)v_1(q) \, dq \, dt = \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q)v_1(q) \, dq \, dt > v(t_2).$$

Since the distribution of ρ does not contain a point mass, $\int_{t_1}^{t_2} \int_{\rho(q) \geq -\phi_c(t)} g(q)v_1(q) \, dq \, dt$ is continuous in c . Thus we must have $c \in (0, 1)$ that satisfies Equation (20). \square

E Optimal Mechanism for Case 1 ($v(t_2) \leq V_L$)

In this section, we derive the optimal mechanism for the first case of Theorem 1. Similar to Section 4.2, we will first prove that our mechanism is feasible. Then we show it achieves the optimal revenue among all feasible mechanisms.

Lemma 9. *The threshold mechanism (π^*, p^*) defined according to $\underline{\phi}^+(t)$ is feasible.*

Proof. Using the characterization of Lemma (4), it suffices to show that the given mechanism satisfies Constraints (15)-(18). Since the ironed lower virtual value function $\underline{\phi}^+(t)$ is monotone non-decreasing, we know that the threshold $\theta_t^* = -\underline{\phi}^+(t)$ is monotone non-increasing in t . This implies that

$$P_{\pi^*}(t) = \int_{q \in Q} \pi^*(q, t) g(q) v_1(q) \, dq = \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q) v_1(q) \, dq$$

is monotone non-decreasing in t since a larger t leads to a smaller integral lower bound, satisfying Constraint (15).

The utility function is, by definition,

$$u(t) = \int_{q \in Q} [g(q) \pi^*(q, t) v(q, t)] \, dq - p^*(t) = \int_{t_1}^t P_{\pi^*}(x) \, dx.$$

which implies $u(t_1) = 0$, and

$$u(t) = u(t_1) + \int_{t_1}^t P_{\pi^*}(x) \, dx,$$

satisfying Constraint (16).

For Constraint (17), we already have $u(t_1) = 0$. Now we prove $u(t_2) \geq v(t_2)$. Lemma 6 shows that the condition $v(t_2) \leq V_L$ is equivalent to $v(t_1) \leq V'_L$. Also, it is easy to see that $V'_L \leq 0$, which implies $v(t_1) \leq 0$. So $\max\{v(t_1), 0\} = 0$, and

$$\begin{aligned} u(t_2) &= \int_{t_1}^{t_2} P_{\pi^*}(x) \, dx = \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q) v_1(q) \, dq \, dx \\ &= \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q) v_1(q) \, dq \, dx + \max\{0, v(t_1)\} \\ &\geq v(t_2) \end{aligned}$$

Finally, we argue that the payment is non-negative, i.e., Constraint (18) is satisfied. By lemma 3, we have for all $t \in T$, $\underline{\phi}^+(t) \leq t$.

Let t' be any number in the interval $[\underline{\phi}^+(t), t]$. Thus

$$\begin{aligned} p^*(t) - p^*(t') &= \int_{q \in Q} [g(q) \pi^*(q, t) v(q, t)] \, dq - \int_{q \in Q} [\pi^*(q, t') g(q) v(q, t')] \, dq - \int_{t'}^t P_{\pi^*}(x) \, dx \\ &= \int_{q: \rho(q) \geq -\underline{\phi}^+(t')} g(q) v(q, t) \, dq - \int_{q: \rho(q) \geq -\underline{\phi}^+(t')} g(q) v(q, t') \, dq - \int_{t'}^t P_{\pi^*}(x) \, dx. \end{aligned}$$

When $\rho(q) \geq -\underline{\phi}^+(t)$, we have $v(q, t') = v_1(q)[t' + \rho(q)] \geq v_1(q)[t' - \underline{\phi}^+(t)] \geq 0$, where the last inequality is because of the choice of t' . So the second term in the above equation satisfies:

$$\begin{aligned} \int_{q: \rho(q) \geq -\underline{\phi}^+(t')} g(q) v(q, t') \, dq &= \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q) v(q, t') \, dq - \int_{q: -\underline{\phi}^+(t) \leq \rho(q) < -\underline{\phi}^+(t')} g(q) v(q, t') \, dq \\ &\leq \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q) v(q, t') \, dq. \end{aligned}$$

Thus,

$$\begin{aligned}
p^*(t) - p^*(t') &\geq \int_{q:\rho(q) \geq -\underline{\phi}^+(t)} g(q)v(q, t) \, dq - \int_{q:\rho(q) \geq -\underline{\phi}^+(t)} g(q)v(q, t') \, dq - \int_{t'}^t P_{\pi^*}(x) \, dx \\
&= \int_{q:\rho(q) \geq -\underline{\phi}^+(t)} g(q)v_1(q)(t - t') \, dq - \int_{t'}^t P_{\pi^*}(x) \, dx \\
&= (t - t')P_{\pi^*}(t) - \int_{t'}^t P_{\pi^*}(x) \, dx \\
&\geq 0,
\end{aligned}$$

where the last inequality is due to the monotonicity of $P_{\pi^*}(t)$.

Therefore, the payment function $p^*(t)$ is monotone non-decreasing in the interval $[\underline{\phi}(t), t]$. Since the set of intervals $\{[\underline{\phi}(t), t] \mid t \in T\}$ covers the interval T , we conclude that $p(t)$ is monotone non-decreasing in T . Therefore, to prove that $p(t) \geq 0$ for all $t \in T$, it suffices to show that $p(t_1) \geq 0$. Indeed, we have

$$p^*(t_1) = \int_{q \in Q} \pi^*(q, t_1)g(q)v(q, t_1) \, dq - u(t_1) = \int_{q:\rho(q) \geq -\underline{\phi}^+(t_1)} g(q)v(q, t_1) \, dq \geq 0.$$

The inequality holds because when $\rho(q) \geq -\underline{\phi}^+(t_1) \geq -t_1$, we get $v(q, t_1) = v_1(q)(t_1 + \rho(q)) \geq 0$. \square

Now we prove that the mechanism defined according to $\underline{\phi}^+(t)$ is optimal, i.e., achieves the maximum possible revenue among all feasible mechanisms.

Lemma 10. *If $v(t_2) \leq V_L$, the threshold mechanism with threshold signaling function $\theta^*(t) = -\underline{\phi}^+(t)$ and the following payment function represents an optimal mechanism:*

$$p^*(t) = \int_{q \in Q} \pi^*(q, t)g(q)v(q, t) \, dq - \int_{t_1}^t \int_{q \in Q} \pi^*(q, x)g(q)v_1(q) \, dq \, dx.$$

where π^* is determined by $\theta^*(t)$ as in Definition 1.

Proof. According to the proof of Lemma 1, the revenue of any feasible mechanism can be written as:

$$REV(\pi, p) = \int_{q \in Q} g(q) \left[\int_{t_1}^{t_2} f(t)\pi(q, t)v_1(q) [\underline{\phi}(t) + \rho(q)] \, dt \right] \, dq - u(t_1).$$

Let $\underline{H}(\cdot)$, $\underline{h}(\cdot)$, $\underline{L}(\cdot)$, and $\underline{l}(\cdot)$ the corresponding function when ironing the virtual value $\underline{\phi}(t)$. We can write the first term of the revenue function as follows:

$$\begin{aligned}
&\int_{q \in Q} \int_{t_1}^{t_2} [\underline{\phi}(t) + \rho(q)] f(t)\pi(q, t)g(q)v_1(q) \, dt \, dq \\
&= \int_{q \in Q} \int_{t_1}^{t_2} [\underline{\phi}^+(t) + \rho(q)] f(t)\pi(q, t)g(q)v_1(q) \, dt \, dq \\
&\quad + \int_{q \in Q} \int_{t_1}^{t_2} [\underline{h}(F(t)) - \underline{l}(F(t))] f(t)\pi(q, t)g(q)v_1(q) \, dt \, dq.
\end{aligned}$$

This is because by definition, $\underline{\phi}^+(t) = \underline{l}(F(t))$ and $\underline{\phi}(t) = \underline{h}(F(t))$. Using integration by parts, we can simplify the second term

$$\begin{aligned} & \int_{q \in Q} \int_{t_1}^{t_2} [\underline{h}(F(t)) - \underline{l}(F(t))] f(t) \pi(q, t) g(q) v_1(q) dt dq \\ &= \int_{t_1}^{t_2} [\underline{h}(F(t)) - \underline{l}(F(t))] P_\pi(t) dF(t) \\ &= [\underline{H}(F(t)) - \underline{L}(F(t))] P_\pi(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} [\underline{H}(F(t)) - \underline{L}(F(t))] dP_\pi(t) \end{aligned}$$

Because \underline{L} is the “convex hull” of \underline{H} on $[0, 1]$, $\underline{L}(0) = \underline{H}(0)$ and $\underline{L}(1) = \underline{H}(1)$. Thus the term $[\underline{H}(F(t)) - \underline{L}(F(t))] P_\pi(t) \Big|_{t_1}^{t_2}$ is simply 0, and we have

$$\begin{aligned} REV(\pi, p) &= \int_{q \in Q} \int_{t_1}^{t_2} [\underline{\phi}^+(t) + \rho(q)] f(t) \pi(q, t) g(q) v_1(q) dt dq \\ &\quad - \int_{t_1}^{t_2} [\underline{H}(F(t)) - \underline{L}(F(t))] dP_\pi(t) - u(t_1) \end{aligned}$$

Now consider mechanism (π^*, p^*) . π^* maximizes the first term since $\pi^*(q, t) = 1, \forall q, t$ with $\rho(q) + \underline{\phi}^+(t) \geq 0$. Also, by definition, we have

$$u(t) = \int_{q \in Q} \pi^*(q, t) g(q) v(q, t) dq - p(t) = \int_{t_1}^t P_{\pi^*}(x) dx.$$

Thus we have $u(t_1) = 0$.

As for the second term, note that $\underline{H}(F(t)) - \underline{L}(F(t)) \geq 0$ by definition, and $dP_\pi(t) \geq 0$ for any feasible mechanism. Thus the second term is always non-negative. However, we claim that with mechanism (π^*, p^*) , this term is actually 0. The only interesting case is when $\underline{H}(F(t)) - \underline{L}(F(t)) > 0$. We will show that $dP_{\pi^*}(t) = 0$. In this case, t must lie in an ironed interval I . Thus $\underline{L}(z)$ is linear in the interval I , where $z = F(t)$. This implies that $\underline{\phi}^+(t) = \underline{l}(z) = \underline{L}'(z)$ is constant. So

$$P_{\pi^*}(t) = \int_{q \in Q} \pi^*(q, t) g(q) v_1(q) dq = \int_{q: \rho(q) \geq -\underline{\phi}^+(t)} g(q) v_1(q) dq$$

is also constant in the interval I , which leads to $dP_{\pi^*}(t)$ being 0.

Therefore, mechanism (π^*, p^*) optimizes all 3 terms in Equation (21) simultaneously, hence optimal. \square

Note that the above derivation of $REV(\pi, p)$ used the equality $u(t) = \int_{t_1}^t P_\pi(x) dx + u(t_1)$ to expand $u(t)$ with t_1 as the reference point. This is also the original Myerson’s approach. This approach works in Myerson’s optimal auction design because there Buyer’s surplus equals Buyer’s utility from participating the mechanism since the only outside option is to not purchase resulting in utility 0. Therefore, in Myerson’s optimal auction design, $u(t_1) \geq 0$ guarantees IR constraint, i.e., $u(t) \geq 0$, for any feasible mechanism. This however, ceases to be true in our setup because $s(t_1) \geq 0$ does not guarantee $s(t_2) \geq 0$. In fact, Lemma 5 shows that $s(t)$ is a concave function with $s(\bar{t})$ as the maximum surplus where \bar{t} is a zero of $v(t)$ function. Nevertheless, we know that the optimal mechanism must satisfy either $s(t_1) = 0$ or $s(t_2) = 0$ since otherwise, we can shift the entire $s(t)$ curve down by a constant — achieved by asking each buyer type to pay the same additional amount — until one of them reaches 0.

F Optimal Mechanism for Case 2 ($v(t_2) \geq V_H$)

In this section, we will discuss the second case of our main result, i.e., when $v(t_2) \geq V_H$. In this case, if we still use t_1 as the reference point and follow the same analysis of Case 1, we will end up having a mechanism with $u(t_2) < v(t_2)$, hence infeasible. To solve this problem, we write the revenue expression $REV(\pi, p)$ using t_2 as the reference point. Although the resulting mechanism looks different, the approach for deriving it is quite similar to that in the proof of Case 1.

We still start with showing the feasibility of the given mechanism (π^*, p^*) .

Lemma 11. *The threshold mechanism (π^*, p^*) defined according to $\bar{\phi}^+(t)$ is feasible.*

Proof. According to Lemma (4), it suffices to show that the given mechanism satisfies Constraints (15)-(18). Since the ironed upper virtual value function $\bar{\phi}^+(t)$ is monotone non-decreasing, we know that the threshold $\theta^*(t) = -\bar{\phi}^+(t)$ is monotone non-increasing in t . This implies that

$$P_{\pi^*}(t) = \int_{q \in Q} \pi^*(q, t) g(q) v_1(q) \, dq = \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q) v_1(q) \, dq$$

is monotone non-decreasing in t since a larger t leads to a larger $-\bar{\phi}^+(t)$ and thus larger integral domain for q . So Constraint (15) is satisfied.

We now prove that (π^*, p^*) satisfies Constraint (16). Plugging the payment function $\pi^*(t)$ into the definition of $u(t)$, we get

$$u(t) = \int_{q \in Q} g(q) \pi^*(q, t) v(q, t) \, dq - p^*(t) = v(t_2) - \int_t^{t_2} P_{\pi^*}(x) \, dx.$$

It is easy to see that $u(t_2) = v(t_2)$, which can be plugged back to the above equality to obtain Constraint (16).

For Constraint (17), we already have $u(t_2) = v(t_2)$. And

$$u(t_1) = v(t_2) - \int_{t_1}^{t_2} P_{\pi^*}(x) \, dx \geq \max\{v(t_1), 0\} + \int_{t_1}^{t_2} P_{\pi^*}(x) \, dx - \int_{t_1}^{t_2} P_{\pi^*}(x) \, dx \geq 0,$$

where the inequality is due to the condition $v(t_2) \geq V_H$.

Finally, we show that the payment $p^*(t)$ is non-negative i.e., $p^*(t)$ satisfies Constraint (18). By lemma 3, we have for all $t \in T$, $t \leq \bar{\phi}^+(t)$.

For any $t > t_1$ and $t' \in [t, \bar{\phi}^+(t)]$, we have

$$\begin{aligned} p^*(t') - p^*(t) &= \int_{q \in Q} \pi^*(q, t') g(q) v(q, t') \, dq - \int_{q \in Q} g(q) \pi^*(q, t) v(q, t) \, dq - \int_t^{t'} P_{\pi^*}(x) \, dx \\ &= \int_{q: \rho(q) \geq -\bar{\phi}^+(t')} g(q) v(q, t') \, dq - \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q) v(q, t) \, dq - \int_t^{t'} P_{\pi^*}(x) \, dx. \end{aligned} \quad (28)$$

Observe that $\bar{\phi}^+(t) \leq \bar{\phi}^+(t')$ since $t \leq t'$. So the first term in the right-hand side can be written as:

$$\int_{q: \rho(q) \geq -\bar{\phi}^+(t')} g(q) v(q, t') \, dq = \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q) v(q, t') \, dq + \int_{q: -\bar{\phi}^+(t') \leq \rho(q) < -\bar{\phi}^+(t)} g(q) v(q, t') \, dq.$$

When $\rho(q) < -\bar{\phi}^+(t)$, we have $v(q, t') = v_1(q)[t' + \rho(q)] \leq v_1(q)[t' - \bar{\phi}^+(t)] \leq 0$, where the inequality is due to the choice of t' . Therefore, the second term in the right-hand side of the above equation is negative. As a result,

$$\int_{q: \rho(q) \geq -\bar{\phi}^+(t')} g(q)v(q, t') dq \leq \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q)v(q, t') dq.$$

Combining with Equation (28), we get

$$\begin{aligned} p^*(t') - p^*(t) &\leq \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q)v(q, t') dq - \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q)v(q, t) dq - \int_t^{t'} P_{\pi^*}(x) dx \\ &= \int_{q: \rho(q) \geq -\bar{\phi}^+(t)} g(q)v_1(q)(t' - t) dq - \int_t^{t'} P_{\pi^*}(x) dx \\ &= (t' - t)P_{\pi^*}(t) - \int_t^{t'} P_{\pi^*}(x) dx \\ &\leq 0. \end{aligned}$$

This shows that $p^*(t)$ is monotone non-increasing in the interval $[t, \bar{\phi}^+(t)]$ for any $t > t_1$. Since set of intervals $\{[t, \bar{\phi}^+(t)] \mid t \in T\}$ covers interval T , we can conclude that $p^*(t)$ is monotone non-increasing in the entire interval T .⁵ Thus, to show that the payment is always non-negative, we only need to prove that $p^*(t_2) \geq 0$. Indeed,

$$\begin{aligned} p^*(t_2) &= \int_{q \in Q} g(q)\pi^*(q, t_2)v(q, t_2) dq - v(t_2) + \int_{t_2}^{t_2} P_{\pi^*}(x) dx \\ &= \int_{q \in Q} g(q)\pi^*(q, t_2)v(q, t_2) dq - \int_{q \in Q} g(q)v(q, t_2) dq \\ &= - \int_{q: \rho(q) < -\bar{\phi}^+(t_2)} g(q)v(q, t_2) dq. \end{aligned}$$

When $\rho(q) < -\bar{\phi}^+(t_2) < -t_2$, we have $v(q, t_2) = v_1(q)[t_2 + \rho(q)] < 0$. Thus $p^*(t_2) \geq 0$. \square

⁵Similar techniques are also used to proved existence of solutions for differential equations.

G Threshold Mechanisms with Partial Recommendations

We assumed that the probability distribution of ρ does not have point masses in the main body of the paper. This is to ensure the existence of the constant c in Case 3 of Theorem 1. But if the distribution of ρ has point masses, such a c may not exist. In this case, we will need to slightly modify our mechanism and incorporate partial recommendations. If such a c does not exist, it must be that both the distributions of ρ and ϕ_c^+ contains point masses, more specifically, the measure of $\{(\rho, t) \mid \rho(q) = \phi_c^+(t) = \zeta\}$ is non-zero for some ζ .

For any $c \in [0, 1]$, let $\phi_c(t) = c\underline{\phi}(t) + (1 - c)\bar{\phi}(t)$ be the mixed virtual value function and

$$Y(c) = \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq dt$$

be a function of c .

We first prove the monotonicity of $Y(c)$. As shown in Lemma 3, $\forall 0 \leq c < c' \leq 1$, we have $\phi_c^+(t) \geq \phi_{c'}^+(t) \forall t$. Thus, when c is increasing, $\phi_c^+(t)$ is (weakly) decreasing and the threshold $-\phi_c^+(t)$ is (weakly) increasing. So the function $Y(c)$ will integrate a non-negative function $g(q)v_1(q)$ over a smaller region of q and is (weakly) decreasing.

Next, we argue that $Y(c)$ is left-continuous. By monotonicity, we know that $Y(c)$ is continuous almost everywhere. For any $c \in (0, 1)$ and any arbitrary small positive ϵ , we have

$$\begin{aligned} & \lim_{\beta \rightarrow c^-} Y(\beta) - Y(c) \\ &= \lim_{\epsilon \rightarrow 0^+} Y(c - \epsilon) - Y(c) \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_{c-\epsilon}^+(t)} g(q) v_1(q) dq dt - \int_{t_1}^{t_2} \int_{q: \rho(q) \geq -\phi_c^+(t)} g(q) v_1(q) dq dt \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{t_1}^{t_2} \int_{q: -\phi_{c-\epsilon}^+(t) \leq \rho(q) < -\phi_c^+(t)} g(q) v_1(q) dq dt \\ &= 0 \end{aligned}$$

where the last equation is because whenever there is a point mass such that the measure of $\{(\rho, t) \mid \rho(q) = \phi_\beta^+(t) = \zeta\}$ is non-zero for some $\beta < c$, we can always increase the lower bound of the integral to exclude this point mass by choosing an ϵ smaller than $c - \beta$. Consequently, we have $\lim_{\beta \rightarrow c^-} Y(\beta) = Y(c)$, so function $Y(c)$ is left continuous on $(0, 1)$.

Now, we are ready to define our signaling function for the case with point mass $\rho(q)$. Since $Y(c)$ is monotone (weakly) decreasing and is left continuous, the following min is well-defined

$$c = \min\{\beta \mid Y(\beta) \geq v(t_2)\}, \quad (29)$$

and moreover we can use binary search to find the c .

Given the above c , we define the following signaling scheme. Define the following constant D ,

$$\begin{aligned} D &= \frac{v(t_2) - (Y(c) - \int_{t_1}^{t_2} \int_{q: \rho(q) = -\phi_c^+(t)} g(q) v_1(q) dq dt)}{\int_{t_1}^{t_2} \int_{q: \rho(q) = -\phi_c^+(t)} g(q) v_1(q) dq dt} \\ &= \frac{v(t_2) - \int_{t_1}^{t_2} \int_{q: \rho(q) > -\phi_c^+(t)} g(q) v_1(q) dq dt}{\int_{t_1}^{t_2} \int_{q: \rho(q) = -\phi_c^+(t)} g(q) v_1(q) dq dt}. \end{aligned}$$

and a corresponding signaling scheme

$$\pi^*(q, t) = \begin{cases} 1 & \text{for all } q, t \text{ such that } \rho(q) > -\phi_c^+(t) \\ D & \text{for all } q, t \text{ such that } \rho(q) = -\phi_c^+(t) \\ 0 & \text{otherwise} \end{cases}$$

This signaling scheme gives rise to a threshold mechanism by using the payment function defined in Theorem 1. Notably, when $\rho(q) = -\phi_c^+(t)$ doesn't have point mass at this c point, D will be 0 due to continuity and this degenerates to the threshold signaling scheme for Case 3 in Theorem 1. The feasibility and the optimality of the above mechanisms follow from the same argument in the proof of Lemma 8 and Proposition 1, essentially because the boundary case of $\rho(q) = -\phi_c^+(t)$ will not affect revenue. We omit details here.